

Correctors justification for a Smoluchowski–Soret–Dufour model posed in perforated domains

Vo Anh Khoa*, and Adrian Muntean†

April 7, 2017

Abstract

We study a coupled thermo-diffusion system that accounts for the dynamics of hot colloids in periodically heterogeneous media. Our model describes the joint evolution of temperature and colloidal concentrations in a saturated porous structure, where the Smoluchowski interactions are responsible for aggregation and fragmentation processes in the presence of Soret-Dufour type effects. Additionally, we allow for deposition and depletion on internal micro-surfaces. In this work, we derive corrector estimates quantifying the rate of convergence of the periodic homogenization limit process performed in [24] via two-scale convergence arguments. The major technical difficulties in the proof are linked to the estimates between nonlinear processes of aggregation and deposition and to the convergence arguments of the *a priori* information of the oscillating weak solutions and cell functions in high dimensions. Essentially, we circumvent the arisen difficulties by a suitable use of the energy method and of fine integral estimates controlling interactions at the level of micro-surfaces.

1 Introduction

Diffusion and heat conduction, taken separately, are well understood processes at a large variety of space scales. However, as soon as diffusion interplays with the conduction of heat, it appears that the structure of the model equations is not so clear as one would expect, especially if one wants to describe settings away from the somewhat better understood thermodynamic equilibrium, where statistical mechanics is the main investigation tool.

Driven by possible applications in the context of efficient drug-delivery and in the design of intelligent packaging materials, we wish to understand mathematically the upscaling of the following basic thermo-diffusion scenario: We look at a population of colloidal particles (monomers) driven by a flux linearly combining Fick and Fourier contributions. We assume that monomers undergo a Smoluchowski-like dynamics producing populations of i -mers that finally meet and travel through a transversal porous membrane. The microscopic boundaries (at the level of the membrane pores) are active in the sense that they host adsorption and desorption of clusters of colloidal particles.

The starting PDE model is formulated in [24] by Krehel and his co-authors. Their thermo-diffusion system is posed in perforated media with uniform periodicity inside the domain. As main outcome, they prove both the global weak solvability of the model as well as the periodic homogenization limit. As byproduct, they also obtain the precise structure of the effective transport parameters. Now, is the moment to: Justify the two-scale asymptotics by proving corrector/error estimates for the homogenization limit for periodic arrangements of membrane pores/microstructures.

*Author for correspondence. Mathematics and Computer Science Division, Gran Sasso Science Institute, L'Aquila, Italy. (khoa.vo@gssi.infn.it, vakhoa.hcmus@gmail.com)

†Department of Mathematics and Computer Science, Karlstad University, Sweden. (adrian.muntean@kau.se)

In our context, the structure of the corrector estimate for the involved concentrations and temperature fields we wish to prove is

$$\begin{aligned} & \|\theta^\varepsilon - \theta_0^\varepsilon\|_{L^2((0,T)\times\Omega^\varepsilon)}^2 + \|u^\varepsilon - u_0^\varepsilon\|_{L^2((0,T)\times\Omega^\varepsilon)}^2 \\ & + \|\nabla(\theta^\varepsilon - \theta_1^\varepsilon)\|_{L^2((0,T)\times\Omega^\varepsilon)}^2 + \|\nabla(u^\varepsilon - u_1^\varepsilon)\|_{L^2((0,T)\times\Omega^\varepsilon)}^2 + \varepsilon \|v^\varepsilon - v_0^\varepsilon\|_{L^2((0,T)\times\Gamma^\varepsilon)}^2 \leq C\varepsilon, \end{aligned} \quad (1.1)$$

where $C > 0$ is a generic constant independent of the choice of the scale parameter $\varepsilon > 0$.

To obtain this corrector estimate, our strategy is to use an energy-like method and macroscopic reconstructions (cf. e.g. [9], but also [10]). This technique basically relies on the choice of test functions able to capture in suitable norms the difference between the micro- and macro-concentrations as well as micro- and macro-temperatures and their transport fluxes. Careful attention needs to be paid to the regularity of the limit solutions as well as of the cell functions involved in the asymptotic procedure; see e.g. [22, 15]. A similar approach has been followed by Eck et al. (cf. e.g. [8, 9]) concerning the upscaling of the phase field model in high contrast regimes. Besides handling new nonlinear terms, the novel aspect in our context is the handling of the errors produced in the upscaling due to micro-surfaces. A similar analysis can be carried over the settings in [4, 34, 36, 14], e.g.

Besides the energy-like approach used here for a periodic homogenization case, powerful contributions can be obtained using variants of the bulk and boundary unfolding operators: see, for instance, [18, 31, 15, 27]. Using somewhat more regularity, high-order corrector estimates can be obtained for semi-linear elliptic systems via an iteration method that uses explicitly the expected structure of the two-scale asymptotic expansion; compare [22, 21]. Settings involving locally-periodic microstructures can be treated as in [28], e.g., while the random case is in most of the cases out of reach; see [23, 33] for some details in this direction.

Having available corrector estimates like (1.1) allows in principle the construction of convergence proofs as well as *a priori* error estimate for MsFEM applied to problems in perforated media like in [7], for instance.

This paper is structured as follows: Section 2 is devoted to the presentation of the Smoluchowski-Sorect-Dufour model posed in a perforated domain. In this section, we also list a couple of preliminary results about the two-scale convergence and compactness arguments and about the weak solvability of both the microscopic and limit models (recalling from [24]). Our main result is Theorem 12, as presented in Section 3. We then introduce the derivation of the difference system resulting from the microscopic problem and the "macroscopic reconstructed" system. On top of that, we prepare in this part a few helpful integral estimates. The proof of Theorem 12 is provided in Section 4. We conclude the paper with the remarks from Section 5.

2 Setting of the problem

2.1 The coupled thermo-diffusion model

2.1.1 A geometrical interpretation of porous medium

Let Ω be a bounded open domain in \mathbb{R}^d ($d \in \{2, 3\}$) with $\partial\Omega \in C^{0,1}$. Without loss of generality, we reduce ourselves to consider Ω as the parallelepiped $(0, a_1) \times \dots \times (0, a_d)$ with $a_i > 0, i \in \{1, \dots, d\}$. Let Y be the representative unit cell defined by

$$Y := \left\{ \sum_{i=1}^d \lambda_i \vec{e}_i : 0 < \lambda_i < 1 \right\},$$

where \vec{e}_i is the i th unit vector in \mathbb{R}^d .

Let Y_0 be an open subset of Y with a Lipschitz boundary $\Gamma = \partial Y_0$ which is divided into two disjoint closed parts Γ_N and Γ_R with a nonzero $(d-1)$ -dimensional measure, i.e. $\Gamma = \Gamma_N \cup \Gamma_R$ with $\Gamma_N \cap \Gamma_R = \emptyset$.

Let $Z \in \mathbb{R}^d$ be a hypercube. Then for $X \subset Z$ we denote by X^k the shifted subset

$$X^k := X + \sum_{i=1}^d k_i \vec{e}_i,$$

where $k = (k_1, \dots, k_d) \in \mathbb{Z}^d$ is a vector of indices.

Assume that a scale factor $\varepsilon > 0$ is given. The pore skeleton is then defined as the union of εY_0^k the ε -homothetic sets of Y_0^k , i.e.

$$\Omega_0^\varepsilon := \bigcup_{k \in \mathbb{Z}^d} \{ \varepsilon Y_0^k : Y_0^k \subset \Omega \}.$$

Thus, the total pore space we have in mind is $\Omega^\varepsilon = \Omega \setminus \Omega_0^\varepsilon$.

Set $Y_1 := Y \setminus \overline{Y_0}$. The unit cell Y is made of two parts including the gas phase Y_1 and the solid phase Y_0 . We denote the total pore surface of the skeleton by $\Gamma^\varepsilon := \partial \Omega_0^\varepsilon$. The pore surface Γ^ε consists of two parts satisfying $\Gamma^\varepsilon = \Gamma_N^\varepsilon \cup \Gamma_R^\varepsilon$ where Γ_N^ε and Γ_R^ε are disjoint closed sets possessing a nonzero $(d-1)$ -dimensional measure. The Neumann boundary Γ_N^ε indicates the insulation for the heat flow, whilst at Γ_R^ε we allow for a flux of mass through a Robin-type condition. The union of the cell regions εY_1^k (without the solid grains εY_0^k) represents the total available space for thermo-diffusion.

In Figure 2.1 and Figure 2.2, we show a admissible 2d domain with microstructures. We let throughout the paper $\mathbf{n} := (n_1, \dots, n_d)$ be the unit outward normal vector on the boundary $\partial \Omega^\varepsilon$. The representation of the periodic geometries is inspired from [20, 22, 34] and references cited therein, but other possibilities exist as well. The practical problem usually delimitates the freedom in choosing the precise structure of Y_0 ; see Figure 2.2 for a couple of options.

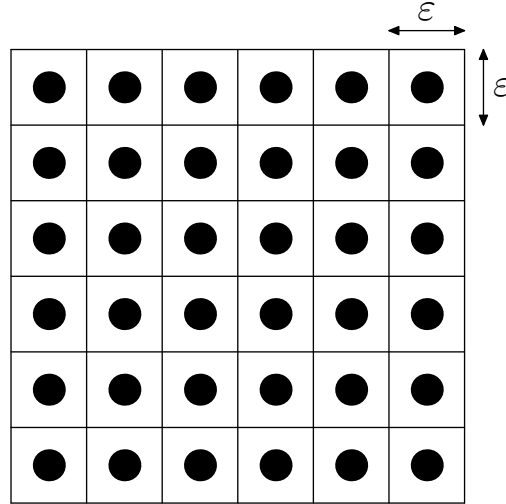


Figure 2.1: An admissible 2d perforated domain.

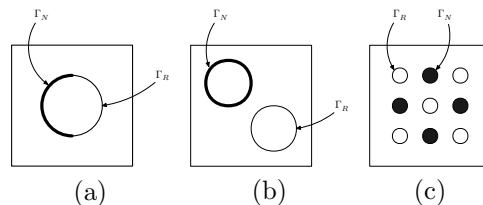


Figure 2.2: Possible choices for Y_0 . The choice of (a) fits to the geometry described in Figure 2.1.

2.1.2 Model description

Before describing the microscopic problem (which we refer to as (P^ε)), we define some useful notation. For $\delta > 0$, let ∇^δ be the so-called mollified gradient

$$\nabla^\delta f(x) := \nabla \left[\int_{B(x, \delta)} J_\delta(x-y) f(y) dy \right],$$

where J_δ is a mollifier (see e.g. [12]) and $B(x, \delta)$ is the ball centered in $x \in \Omega$ with radius δ . The radius δ is assumed to be an ε -independent constant.

We denote by $x \in \Omega^\varepsilon$ the macroscopic variable and by $y = x/\varepsilon$ the microscopic variable representing fast variations at the microscopic geometry. With this convention, we write

$$\kappa^\varepsilon(x) = \kappa\left(\frac{x}{\varepsilon}\right) = \kappa(y).$$

The same convention applies to all the other oscillating coefficients involved our problem.

We denote by $\mathcal{A}_\mathbb{T}^\varepsilon$ the second-order elliptic operator in divergence form with rapidly oscillating coefficients, i.e.

$$\mathcal{A}_\mathbb{T}^\varepsilon := \nabla \cdot \left(-\mathbb{T}\left(\frac{x}{\varepsilon}\right) \nabla \right) = \frac{\partial}{\partial x_i} \left[-\tau_{ij}^{\alpha\beta}\left(\frac{x}{\varepsilon}\right) \frac{\partial}{\partial x_j} \right]. \quad (2.1)$$

Concerning the structure of $\mathcal{A}_\mathbb{T}^\varepsilon$, we assume that for all $y \in Y$, $\mathbb{T}(y) = \left(\tau_{ij}^{\alpha\beta}(y) \right) : \mathbb{R}^d \rightarrow \mathbb{R}^{m^2 \times d^2}$ for $1 \leq i, j \leq d, 1 \leq \alpha, \beta \leq m$ is a second-order tensor that depends on the position vector y and satisfies a uniform (in ε) ellipticity condition. Depending on the situation, we have either \mathbb{T} is the tensor κ (heat conductivity) or the tensor d_i (diffusion coefficients). Note that $m \geq 1$ denotes the number of balance equations in the system.

In this framework, we consider that maximum $N > 2$ colloidal species are involved in the thermo-diffusion process. We denote by $(\theta^\varepsilon, u_i^\varepsilon, v_i^\varepsilon)$ for $i \in \{1, \dots, N\}$ the triplet of real-valued solutions of our thermo-diffusion model, i.e. a system of coupled ordinary differential equations with semi-linear parabolic equations for the evolution of temperature and colloid concentrations. Denote by $u^\varepsilon := (u_1^\varepsilon, \dots, u_N^\varepsilon)$ the vector of all active colloidal concentrations u_i^ε . We assume that these species obey the population balance equation as postulated by Smoluchowski in [37], i.e.

$$R_i(s) := \frac{1}{2} \sum_{k+j=i} \beta_{kj} s_k s_j - \sum_{j=1}^N \beta_{ij} s_i s_j, \quad (\text{with } R_i : \mathbb{R}^N \rightarrow \mathbb{R}, i \in \{1, \dots, N\})$$

theoretically representing a quadratic-like rate of change of s_i . The presence of coagulation coefficients $\beta_{ij} > 0$ accounts for the rate aggregation and fragmentation between populations of particles of size i and j . For further modeling details, we refer the reader to [11, 16, 17] and [25], e.g.

We denote the parabolic cylinders as $Q_T^\varepsilon := (0, T) \times \Omega^\varepsilon$ and $Q_T := (0, T) \times \Omega$. Now, we detail the structure of our microscopic problem (P^ε) . For $i \in \{1, \dots, N\}$, we consider the following coupled thermo-diffusion system:

$$\partial_t \theta^\varepsilon + \mathcal{A}_\kappa^\varepsilon \theta^\varepsilon = \tau^\varepsilon \sum_{i=1}^N \nabla^\delta u_i^\varepsilon \cdot \nabla \theta^\varepsilon \quad \text{in } Q_T^\varepsilon, \quad (2.2)$$

$$\partial_t u_i^\varepsilon + \mathcal{A}_{d_i}^\varepsilon u_i^\varepsilon = \rho_i^\varepsilon \nabla^\delta \theta^\varepsilon \cdot \nabla u_i^\varepsilon + R_i(u^\varepsilon) \quad \text{in } Q_T^\varepsilon, \quad (2.3)$$

$$\partial_t v_i^\varepsilon = a_i^\varepsilon u_i^\varepsilon - b_i^\varepsilon v_i^\varepsilon \quad \text{on } (0, T) \times \Gamma^\varepsilon, \quad (2.4)$$

subject to the boundary conditions

$$-\kappa^\varepsilon \nabla \theta^\varepsilon \cdot \mathbf{n} = 0 \quad \text{on } (0, T) \times \Gamma_N^\varepsilon, \quad (2.5)$$

$$-\kappa^\varepsilon \nabla \theta^\varepsilon \cdot \mathbf{n} = \varepsilon g_0^\varepsilon \theta^\varepsilon \quad \text{on } (0, T) \times \Gamma_R^\varepsilon, \quad (2.6)$$

$$-\kappa^\varepsilon \nabla \theta^\varepsilon \cdot \mathbf{n} = 0 \quad \text{on } (0, T) \times \partial\Omega, \quad (2.7)$$

$$-d_i^\varepsilon \nabla u_i^\varepsilon \cdot \mathbf{n} = \varepsilon (a_i^\varepsilon u_i^\varepsilon - b_i^\varepsilon v_i^\varepsilon) \quad \text{on } (0, T) \times \Gamma^\varepsilon, \quad (2.8)$$

$$-d_i^\varepsilon \nabla u_i^\varepsilon \cdot \mathbf{n} = 0 \quad \text{on } (0, T) \times \partial\Omega, \quad (2.9)$$

and the initial data

$$\theta^\varepsilon(0, x) = \theta^{\varepsilon,0}(x) \quad \text{for } x \in \Omega^\varepsilon. \quad (2.10)$$

$$u_i^\varepsilon(0, x) = u_i^{\varepsilon,0}(x) \quad \text{for } x \in \Omega^\varepsilon, \quad (2.11)$$

$$v_i^\varepsilon(0, x) = v_i^{\varepsilon,0}(x) \quad \text{for } x \in \Gamma^\varepsilon. \quad (2.12)$$

(2.2)-(2.12) form our microscopic problem (P^ε) .

Table 1: Physical parameters of the microscopic problem (P^ε) .

κ^ε	heat conductivity (tensor)
τ^ε	Soret coefficient (tensor)
g_0^ε	heat absorption (scalar)
d_i^ε	diffusion coefficients (tensor)
ρ_i^ε	Dufour coefficients (tensor)
$a_i^\varepsilon, b_i^\varepsilon$	deposition rate coefficients (scalars)

Remark 1. Our thermo-diffusion system is made of $N + 1$ equations where the short-hand explanation for physical parameters in this model can be found in Table 1. Physically, equation (2.2) describes the changes of the temperature θ^ε in Ω^ε according to a heat conduction equation with a production term depending on $\nabla^\delta u_i^\varepsilon$, whilst the colloidal concentration u_i^ε is assumed to satisfy N reaction-diffusion like equations given by (2.3) with a chemical reaction term depending on $\nabla^\delta \theta^\varepsilon$. This type of special right-hand sides is mimicking the so-called Soret and Dufour effects. In (2.8), v_i^ε denotes the mass of the deposited species on the boundary of the pore skeleton Γ^ε . These quantities are also supposed to satisfy the following ordinary differential equations (2.4).

We make use of the following assumptions:

(A₁) The coefficients $\kappa^\varepsilon, \tau^\varepsilon, d_i^\varepsilon, \rho_i^\varepsilon \in [H_+^1(\Omega^\varepsilon)]^{d^2} \cap [L_+^\infty(\Omega^\varepsilon)]^{d^2}$, $g_0^\varepsilon \in L_+^\infty(\Gamma_R^\varepsilon)$ and $a_i^\varepsilon, b_i^\varepsilon \in L_+^\infty(\Gamma^\varepsilon)$ are Y -periodic. Also, there exist positive constants $\kappa_{\min}, \kappa_{\max}, \tau_{\min}, \tau_{\max}, d_{\min}, d_{\max}, \rho_{\min}, \rho_{\max}, a_{\min}, a_{\max}, b_{\min}, b_{\max}$ such that $\kappa_{\min} \leq \kappa_{jk} \leq \kappa_{\max}$, $\tau_{\min} \leq \tau_{jk} \leq \tau_{\max}$, $d_{\min} \leq d_i^{jk} \leq d_{\max}$, $\rho_{\min} \leq \rho_i^{jk} \leq \rho_{\max}$, $a_{\min} \leq a_i^\varepsilon \leq a_{\max}$, $b_{\min} \leq b_i^\varepsilon \leq b_{\max}$ for $i \in \{1, \dots, N\}$ and $j, k \in \{1, \dots, d\}$. Furthermore, there also exist positive constants α_i for $i \in \{0, \dots, N\}$ such that

$$\kappa_{jk}(y) \xi_j \xi_k \geq \alpha_0 |\xi|^2 \quad \text{and} \quad d_i^{jk}(y) \xi_j \xi_k \geq \alpha_i |\xi|^2 \quad \text{for any } \xi \in \mathbb{R}^d, i \in \{1, \dots, N\}, j \text{ and } k \in \{1, \dots, d\}$$

to guarantee the ellipticity of the operators $\mathcal{A}_\kappa^\varepsilon$ and $\mathcal{A}_{d_i}^\varepsilon$.

(A₂) The initial conditions satisfy $\theta^{\varepsilon,0} \in L_+^\infty(\Omega^\varepsilon) \cap H^1(\Omega^\varepsilon)$, $u_i^{\varepsilon,0} \in L_+^\infty(\Omega^\varepsilon) \cap H^1(\Omega^\varepsilon)$, $v_i^{\varepsilon,0} \in L_+^\infty(\Gamma^\varepsilon)$ for $i \in \{1, \dots, N\}$, such that we can find $C_0 > 0$ satisfying

$$\|\theta^{\varepsilon,0}\|_{H^1(\Omega^\varepsilon)} + \sum_{i=1}^N \left(\|u_i^{\varepsilon,0}\|_{H^1(\Omega^\varepsilon)} + \|v_i^{\varepsilon,0}\|_{L^\infty(\Gamma^\varepsilon)} \right) \leq C_0,$$

where C_0 is independent of the choice of ε .

Remark 2. By the definitions of $\kappa, \tau, d_i, \rho_i$ and (A₁), there exist positive constants that bound from below and above these coefficients on Y for each choice of ε .

Unless otherwise specified, all the constants C are independent of the homogenization parameter ε , but the precise values may differ from line to line or even within a single chain of estimates. Throughout this paper, we use the superscript ε to emphasize the dependence on the heterogeneity of the material characterized by the homogenization parameter ε . In the sequel, we use dS_ε as a shorthand for ndS_ε where S_ε can be viewed as a common notation for a boundary of any surface. Moreover, the notation $|\cdot|$ for a domain indicates in this work the volume of that domain.

2.2 Preliminary results

In this subsection, we present the definition of two-scale convergence as well as its compactness arguments (cf. [2, 30]) together with the fact already known concerning the weak solvability and periodic homogenization of (P^ε) .

Definition 3. Two-scale convergence

Let (u^ε) be a sequence of functions in $L^2(0, T; L^2(\Omega))$ with Ω being an open set in \mathbb{R}^d , then it two-scale converges to a unique function $u^0 \in L^2((0, T) \times \Omega \times Y)$, denoted by $u^\varepsilon \xrightarrow{2} u^0$, if for any $\varphi \in C_0^\infty((0, T) \times \Omega; C_\#^\infty(Y))$ we have

$$\lim_{\varepsilon \rightarrow 0} \int_0^T \int_\Omega u^\varepsilon(t, x) \varphi\left(t, x, \frac{x}{\varepsilon}\right) dx dt = \frac{1}{|Y|} \int_0^T \int_\Omega \int_Y u^0(t, x, y) \varphi(t, x, y) dy dx dt.$$

Theorem 4. Two-scale compactness

- Let (u^ε) be a bounded sequence in $L^2((0, T) \times \Omega)$. Then there exists a function $u^0 \in L^2((0, T) \times \Omega \times Y)$ such that, up to a subsequence, u^ε two-scale converges to u^0 .
- Let (u^ε) be a bounded sequence in $L^2(0, T; H^1(\Omega))$, then up to a subsequence, we have the two-scale convergence in gradient $\nabla u^\varepsilon \xrightarrow{2} \nabla_x u^0 + \nabla_y u^1$ for $u^0 \in L^2(0, T; H^1(\Omega))$ and $u^1 \in L^2((0, T) \times \Omega; H_\#^1(Y)/\mathbb{R})$.

Remark 5. The concepts of two-scale convergence and compactness for ε -periodic hypersurfaces were originally introduced in [29, 3] and have been used in [14, 24]. For brevity, let (u^ε) be a sequence of functions in $L^2(0, T; L^2(\Gamma^\varepsilon))$. We say u^ε two-scale converges to a limit u^0 in $L^2((0, T) \times \Omega \times \Gamma)$ with $\Gamma = \partial\Omega$ if for any $\varphi \in C_0^\infty((0, T) \times \Omega; C_\#^\infty(\Gamma))$ we have

$$\lim_{\varepsilon \rightarrow 0} \int_0^T \int_{\Gamma^\varepsilon} \varepsilon u^\varepsilon(t, x) \varphi\left(t, x, \frac{x}{\varepsilon}\right) dx dt = \frac{1}{|Y|} \int_0^T \int_\Omega \int_\Gamma u^0(t, x, y) \varphi(t, x, y) d\sigma(y) dx dt.$$

Thereby, we obtain the two-scale compactness on surfaces that for each bounded sequence (u^ε) in $L^2(0, T; L^2(\Gamma^\varepsilon))$, one can extract a subsequence which two-scale converges to $u^0 \in L^2((0, T) \times \Omega \times \Gamma)$. Furthermore, if (u^ε) is bounded in $L^\infty(0, T; L^\infty(\Gamma^\varepsilon))$, it then two-scale converges to a limit function $u^0 \in L^\infty((0, T) \times \Omega \times \Gamma)$.

It is important to note that, for our choice of Y_0 , the interior extension from $H^1(\Omega^\varepsilon)$ into $H^1(\Omega)$ exists with extension constants independent of ε (see [20, Lemma 5] and [6, Theorem 2.10]).

Definition 6. The weak formulation of (P^ε)

For $i \in \{1, \dots, N\}$, the triplet $(\theta^\varepsilon, u_i^\varepsilon, v_i^\varepsilon)$ satisfying

$$\begin{aligned} \theta^\varepsilon, u_i^\varepsilon &\in H^1(0, T; L^2(\Omega^\varepsilon)) \cap L^\infty(0, T; H^1(\Omega^\varepsilon)) \cap L^\infty((0, T) \times \Omega^\varepsilon), \\ v_i^\varepsilon &\in H^1(0, T; L^2(\Gamma^\varepsilon)) \cap L^\infty((0, T) \times \Gamma^\varepsilon). \end{aligned}$$

is a weak solution to (P^ε) provided that

$$\left\{ \begin{array}{l} \int_{\Omega^\varepsilon} \partial_t \theta^\varepsilon \varphi dx + \int_{\Omega^\varepsilon} \kappa^\varepsilon \nabla \theta^\varepsilon \cdot \nabla \varphi dx + \varepsilon \int_{\Gamma_R^\varepsilon} g_0 \theta^\varepsilon \varphi dS_\varepsilon = \int_{\Omega^\varepsilon} \tau^\varepsilon \sum_{i=1}^N \nabla^\delta u_i^\varepsilon \cdot \nabla \theta^\varepsilon \varphi dx, \\ \int_{\Omega^\varepsilon} \partial_t u_i^\varepsilon \phi_i dx + \int_{\Omega^\varepsilon} d_i^\varepsilon \nabla u_i^\varepsilon \cdot \nabla \phi_i dx + \varepsilon \int_{\Gamma^\varepsilon} (a_i^\varepsilon u_i^\varepsilon - b_i^\varepsilon v_i^\varepsilon) \phi_i dS_\varepsilon \\ \quad = \int_{\Omega^\varepsilon} R_i(u^\varepsilon) \phi_i dx + \int_{\Omega^\varepsilon} \rho_i^\varepsilon \nabla^\delta \theta^\varepsilon \cdot \nabla u_i^\varepsilon \phi_i dx, \\ \varepsilon \int_{\Gamma^\varepsilon} \partial_t v_i^\varepsilon \psi_i dS_\varepsilon = \varepsilon \int_{\Gamma^\varepsilon} (a_i^\varepsilon u_i^\varepsilon - b_i^\varepsilon v_i^\varepsilon) \psi_i dS_\varepsilon, \end{array} \right. \quad (2.13)$$

for all $(\varphi, \phi_i, \psi_i) \in H^1(\Omega^\varepsilon) \times H^1(\Omega^\varepsilon) \times L^2(\Gamma^\varepsilon)$.

Theorem 7. Well-posedness and Positivity of solution

Assume (A_1) - (A_2) and $i \in \{1, \dots, N\}$. The microscopic problem (P^ε) admits a unique solution $(\theta^\varepsilon, u_i^\varepsilon, v_i^\varepsilon)$ in the sense of Definition 6, belonging to

$$K(T, M) := \{z \in L^2((0, T) \times \Omega^\varepsilon) : |z| \leq M \text{ a.e. in } (0, T) \times \Omega^\varepsilon\}$$

for some $M > 0$. Additionally,

$$\begin{aligned} \theta^\varepsilon, u_i^\varepsilon &\in H^1(0, T; L^2(\Omega^\varepsilon)) \cap L^\infty(0, T; H^1(\Omega^\varepsilon)) \cap L^\infty((0, T) \times \Omega^\varepsilon), \\ v_i^\varepsilon &\in H^1(0, T; L^2(\Gamma^\varepsilon)) \cap L^\infty((0, T) \times \Gamma^\varepsilon). \end{aligned}$$

Furthermore, this triplet $(\theta^\varepsilon, u_i^\varepsilon, v_i^\varepsilon)$ is positive and the following energy estimates hold

$$\begin{aligned} \kappa_{\min} \|\nabla \theta^\varepsilon(t)\|_{L^2(\Omega^\varepsilon)}^2 + \int_0^t \|\partial_t \theta^\varepsilon(t)\|_{L^2(\Omega^\varepsilon)}^2 dt &\leq C, \\ \|\nabla u_i^\varepsilon(t)\|_{L^2(\Omega^\varepsilon)}^2 + \int_0^T \left(\|\partial_t u_i^\varepsilon(t)\|_{L^2(\Omega^\varepsilon)}^2 + \|\partial_t v_i^\varepsilon(t)\|_{L^2(\Gamma^\varepsilon)}^2 \right) dt &\leq C \quad \text{for a.e. } t \in (0, T]. \end{aligned}$$

We denote by (P^0) the strong formulation of the macroscopic (limit) problem. We introduce below the limit problem whose precise structure has been obtained via a two-scale convergence procedure in [24].

Theorem 8. Strong formulation of the macroscopic problem – (P^0)

Assume (A_1) - (A_2) . For $i \in \{1, \dots, N\}$, the triplet (θ^0, u_i^0, v_i^0) of limit solutions $(\theta^\varepsilon, u_i^\varepsilon, v_i^\varepsilon)$ to (P^ε) in the sense of Definition 6 satisfies the following macroscopic system

$$\partial_t \theta^0 + \nabla \cdot (-\mathbb{K} \nabla \theta^0) + g_0 \frac{|\Gamma_R|}{|Y_1|} \theta^0 = \sum_{i=1}^N (\mathbb{T}^i \nabla^\delta u_i^0) \cdot \nabla \theta^0 \quad \text{in } Q_T, \quad (2.14)$$

$$\partial_t u_i^0 + \nabla \cdot (-\mathbb{D}^i \nabla u_i^0) + A_i u_i^0 - B_i v_i^0 = (\mathbb{F}^i \nabla u_i^0) \cdot \nabla^\delta \theta^0 + R_i(u^0) \quad \text{in } Q_T, \quad (2.15)$$

subject to the boundary conditions

$$-\mathbb{K} \nabla \theta^0 \cdot n = 0 \quad \text{on } (0, T) \times \partial\Omega, \quad (2.16)$$

$$-\mathbb{D}^i \nabla u_i^0 \cdot n = 0 \quad \text{on } (0, T) \times \partial\Omega, \quad (2.17)$$

and associated with the ordinary differential equations

$$\partial_t v_i^0 = A_i u_i^0 - B_i v_i^0 \quad \text{in } Q_T, \quad (2.18)$$

where we have denoted by $\mathbb{K} = K_0 \mathbb{I} + (K_{ij})_{ij}$, $\mathbb{T}^i = T_0^i \mathbb{I} + (T_{jk}^i)_{jk}$, $\mathbb{D}^i = D_i \mathbb{I} + \mathbb{D}_0^i$, $\mathbb{F}^i = F_i \mathbb{I} + \mathbb{F}_0^i$ for $j, k \in \{1, \dots, d\}$ with \mathbb{I} standing for the identity matrix and the quantities $K_0, K_{ij}, T_0^i, T_{jk}^i, D_i, \mathbb{D}_0^i, F_i, \mathbb{F}^i, A_i, B_i$ being effective constants corresponding, respectively, to the oscillating coefficients and defined in (2.23)-(2.27).

Furthermore, the initial conditions are provided by

$$\theta^0(t=0) = \theta^{0,0} \quad \text{in } \bar{\Omega}, \quad (2.19)$$

$$u_i^0(t=0) = u_i^{0,0} \quad \text{in } \bar{\Omega}, \quad (2.20)$$

$$v_i^0(t=0) = v_i^{0,0} \quad \text{on } \Gamma. \quad (2.21)$$

Theorem 9. The weak formulation of (P^0)

Assume (A_1) -(A_2) and take $i \in \{1, \dots, N\}$, the triplet (θ^0, u_i^0, v_i^0) satisfying

$$\theta^0, u_i^0 \in H^1(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)) \cap L^\infty((0, T) \times \Omega),$$

$$v_i^0 \in H^1(0, T; L^2(\Omega)) \cap L^\infty((0, T) \times \Omega),$$

is a weak solution to (P^0) provided that

$$\left\{ \begin{array}{l} \int_{\Omega} \partial_t \theta^0 \varphi dx + \int_{\Omega} \mathbb{K} \nabla \theta^0 \cdot \nabla \varphi dx + g_0 \frac{|\Gamma_R|}{|Y_1|} \int_{\Omega} \theta^0 \varphi dx = \int_{\Omega} \sum_{i=1}^N (\mathbb{T}^i \nabla^\delta u_i^0) \cdot \nabla \theta^0 \varphi dx, \\ \int_{\Omega} \partial_t u_i^0 \phi_i dx + \int_{\Omega} \mathbb{D}^i \nabla u_i^0 \cdot \nabla \phi_i dx + \int_{\Omega} (A_i u_i^0 - B_i v_i^0) \phi_i dx \\ \quad = \int_{\Omega} (\mathbb{F}^i \nabla u_i^0) \cdot \nabla^\delta \theta^0 \phi_i dx + \int_{\Omega} R_i(u^0) \phi_i dx, \\ \int_{\Omega} \partial_t v_i^0 \psi_i dx = \int_{\Omega} (A_i u_i^0 - B_i v_i^0) \psi_i dx, \end{array} \right. \quad (2.22)$$

hold for all $(\varphi, \phi_i, \psi_i) \in C^\infty(\Omega) \times C^\infty(\Omega) \times C^\infty(\Omega)$.

For $i \in \{1, \dots, N\}$ and $j, k \in \{1, \dots, d\}$, the effective constants in Theorem 8 are defined, as follows:

$$K_0 := \frac{1}{|Y_1|} \int_{Y_1} \kappa(y) dy, \quad K_{ij} := \frac{1}{|Y_1|} \int_{Y_1} \kappa(y) \frac{\partial \bar{\theta}^j}{\partial y_i} dy, \quad (2.23)$$

$$T_0^i := \frac{1}{|Y_1|} \int_{Y_1} \tau_i(y) dy, \quad T_{jk}^i := \frac{1}{|Y_1|} \int_{Y_1} \tau_i(y) \frac{\partial \bar{\theta}^j}{\partial y_i} dy, \quad (2.24)$$

$$D_i := \frac{1}{|Y_1|} \int_{Y_1} d_i(y) dy, \quad \mathbb{D}_0^i := \left(\frac{1}{|Y_1|} \int_{Y_1} d_i(y) \frac{\partial \bar{u}_i^j}{\partial y_k} dy \right)_{jk}, \quad (2.25)$$

$$F_i := \frac{1}{|Y_1|} \int_{Y_1} \rho_i(y) dy, \quad \mathbb{F}^i := \left(\frac{1}{|Y_1|} \int_{Y_1} \rho_i(y) \frac{\partial \bar{u}_i^j}{\partial y_k} dy \right)_{jk}, \quad (2.26)$$

$$A_i := \frac{1}{|Y_1|} \int_{\partial Y_0} a_i dy, \quad B_i := \frac{1}{|Y_1|} \int_{\partial Y_0} b_i dy. \quad (2.27)$$

Hereby, the functions $\bar{\theta}$ and \bar{u}_i linearly formulate the limit functions θ^1 and u_i^1 by $\theta^1 := \bar{\theta} \cdot \nabla_x \theta^0 = \sum_{j=1}^d \partial_{x_j} \theta^0 \bar{\theta}^j$ and $u_i^1 := \bar{u}_i \cdot \nabla_x u_i^0 = \sum_{j=1}^d \partial_{x_j} u_i^0 \bar{u}_i^j$ for $i \in \{1, \dots, N\}$. Moreover, they solve, respectively, the cell problems introduced in the following Theorem.

Theorem 10. The cell problems

Assume (A_1) holds. The limit functions θ^1 and u_i^1 defined as above solve the following cell problems:

$$\begin{cases} \nabla_y \cdot (-\kappa(y) \nabla_y \bar{\theta}^j(x, y)) = \nabla_y \cdot (\kappa n_j) & \text{in } Y_1, \\ -\kappa(y) \nabla_y \bar{\theta}^j \cdot n = \kappa n_j & \text{on } \partial Y_0, \\ \bar{\theta}^j \text{ is } Y\text{-periodic}, \end{cases} \quad (2.28)$$

$$\begin{cases} \nabla_y \cdot (-d_i(y) \nabla_y \bar{u}_i^j(x, y)) = \nabla_y \cdot (d_i n_j) & \text{in } Y_1, \\ -d_i(y) \nabla_y \bar{u}_i^j \cdot n = d_i n_j & \text{on } \partial Y_0, \\ \bar{u}_i^j \text{ is } Y\text{-periodic}, \end{cases} \quad (2.29)$$

where n_j is the j th unit vector of \mathbb{R}^d and $i \in \{1, \dots, N\}, j \in \{1, \dots, d\}$. Furthermore,

- (i) If $\kappa, d_i \in [H^1(\bar{Y}_1)]^{d^2}$ are Lipschitz continuous, the system (2.28)-(2.29) admits a unique solution $(\bar{\theta}^j, \bar{u}_i^j) \in H_{loc}^2(Y_1) \times H_{loc}^2(Y_1)$;
- (ii) If $k, d_i \in [H^1(Y_1)]^{d^2} \cap [H^{-\frac{1}{2}+s}(\partial Y_0)]^{d^2}$ for every $s \in (-\frac{1}{2}, \frac{1}{2})$ are Lipschitz continuous, the system (2.28)-(2.29) admits a unique solution $(\bar{\theta}^j, \bar{u}_i^j) \in H^{1+s}(Y_1) \times H^{1+s}(Y_1)$.

The weak solvability of the cell problems (2.28) and (2.29) shall be further discussed in the proof of our main result – Theorem 12. To derive our corrector estimates, we need a number of elementary inequalities.

- For all $1 \leq p \leq \infty$, the following estimates hold:

$$\|\nabla^\delta f \cdot g\|_{L^p(\Omega^\varepsilon)} \leq C_\delta \|f\|_{L^\infty(\Omega^\varepsilon)} \|g\|_{[L^p(\Omega^\varepsilon)]^d} \quad \text{for } f \in L^\infty(\Omega^\varepsilon), g \in [L^p(\Omega^\varepsilon)]^d, \quad (2.30)$$

$$\|\nabla^\delta f\|_{L^p(\Omega^\varepsilon)} \leq C_\delta \|f\|_{L^2(\Omega^\varepsilon)} \quad \text{for } f \in L^2(\Omega^\varepsilon), \quad (2.31)$$

where $C > 0$ depends only on δ . See [24], e.g., for a proof of (2.30) and (2.31).

- To estimate the correctors for both the temperature θ^ε and colloidal concentrations u_i^ε , we consider the real-valued cut-off function $m^\varepsilon \in C_0^1(\Omega)$ satisfying $0 \leq m^\varepsilon \leq 1$, $\varepsilon |\nabla m^\varepsilon| \leq C$, and $m^\varepsilon = 1$ on $\{x \in \Omega : \text{dist}(x, \Gamma) \geq \varepsilon\}$. Furthermore, one can prove that

$$\|1 - m^\varepsilon\|_{L^2(\Omega^\varepsilon)} \leq C\varepsilon^{1/2}, \quad \varepsilon \|\nabla m^\varepsilon\|_{L^2(\Omega^\varepsilon)} \leq C\varepsilon^{1/2}. \quad (2.32)$$

- (A Young-type inequality) Let $\delta > 0$ and $a, b \geq 0$ be arbitrarily real numbers and take $q, q' > 1$ real constants that are Hölder conjugates of each other. Then the following inequality holds

$$ab \leq \frac{1}{q} \delta^q a^q + \frac{1}{q'} \delta^{-q'} b^{q'}. \quad (2.33)$$

- (Trace inequality for ε -dependent hypersurfaces Γ^ε) Let Γ^ε be as in Subsection 2.1.1. For $\varphi^\varepsilon \in H^1(\Omega^\varepsilon)$, there exists a constant $C > 0$ (independent of ε) such that

$$\varepsilon \|\varphi^\varepsilon\|_{L^2(\Gamma^\varepsilon)}^2 \leq C \left(\|\varphi^\varepsilon\|_{L^2(\Omega^\varepsilon)}^2 + \varepsilon^2 \|\nabla \varphi^\varepsilon\|_{L^2(\Omega^\varepsilon)}^2 \right). \quad (2.34)$$

The proof of (2.34) can be found in [20, Lemma 3].

Theorem 11. Existence and uniqueness results for (P^0) Assume (A_1) – (A_2) . For $i \in \{1, \dots, N\}$, the macroscopic problem (P^0) admits a unique (local) weak solution in $L^2((0, T) \times \Omega)$.

Proof. Due to the homogenization limit results in [24, Lemma 4.3], the existence of the triplet (θ^0, u_i^0, v_i^0) in Theorem 9 is guaranteed. The contraction of these functions in a closed subspace of $[L^2((0, T) \times \Omega)]^{N+2}$ can be proved concisely by a linearization argument. The proof can be sketched as follows: We define

$$K_1(M, T) := \{z \in L^2((0, T) \times \Omega) : |z| \leq M \text{ a.e. in } Q_T\}.$$

For $i \in \{1, \dots, N\}$, let $\theta^{0,1}, u_i^{0,1}, v_i^{0,1} \in K_1(M_1, T_1)$ and $\theta^{0,2}, u_i^{0,2}, v_i^{0,2} \in K_1(M_2, T_2)$ be two pairs of (weak) solutions of the macro system. By choosing $T = \min\{T_1, T_2\}$ and $M = 2 \max\{M_1, M_2\}$ and suitable test functions φ, ϕ_i, ψ_i in (2.22), we get $d(\theta^0) := \theta^{0,1} - \theta^{0,2}, d(u_i^0) := u_i^{0,1} - u_i^{0,2}, d(v_i^0) := v_i^{0,1} - v_i^{0,2} \in K_1(M, T)$, which satisfy the following equalities:

$$\begin{aligned} & \frac{1}{2} \partial_t \|d(\theta^0)\|_{L^2(\Omega)}^2 + \mathbb{K} \|\nabla d(\theta^0)\|_{L^2(\Omega)}^2 + g_0 \frac{|\Gamma_R|}{|\Upsilon_1|} \|d(\theta^0)\|_{L^2(\Omega)}^2 \\ &= \int_{\Omega} \sum_{i=1}^N ((\mathbb{T}^i \nabla^\delta u_i^{0,1}) \cdot \nabla \theta^{0,1} - (\mathbb{T}^i \nabla^\delta u_i^{0,2}) \cdot \nabla \theta^{0,2}) d(\theta^0) dx, \end{aligned} \quad (2.35)$$

$$\begin{aligned} & \frac{1}{2} \partial_t \|d(u_i^0)\|_{L^2(\Omega)}^2 + \mathbb{D}^i \|\nabla d(u_i^0)\|_{L^2(\Omega)}^2 + A_i \|d(u_i^0)\|_{L^2(\Omega)}^2 - \int_{\Omega} B_i d(v_i^0) d(u_i^0) dx \\ &= \int_{\Omega} ((\mathbb{F}^i \nabla u_i^{0,1}) \cdot \nabla^\delta \theta^{0,1} - (\mathbb{F}^i \nabla u_i^{0,2}) \cdot \nabla^\delta \theta^{0,2}) d(u_i^0) dx \\ &+ \int_{\Omega} (R_i(u_i^{0,1}) - R_i(u_i^{0,2})) d(u_i^0) dx, \end{aligned} \quad (2.36)$$

$$\frac{1}{2} \partial_t \|d(v_i^0)\|_{L^2(\Omega)}^2 + B_i \|d(v_i^0)\|_{L^2(\Omega)}^2 = \int_{\Omega} A_i d(u_i^0) d(v_i^0) dx.$$

Then, with the help of the estimates (2.30)–(2.31) and the Young-type inequality (2.33) under a suitable choice of a pair (δ, q, q') to get rid of the gradient norms $\|\nabla d(\theta^0)\|_{L^2(\Omega)}^2$ and $\|\nabla d(u_i^0)\|_{L^2(\Omega)}^2$ on the left-hand side of (2.35)–(2.36), one can find a constant $C(M, \delta) > 0$ such that for all $i \in \{1, \dots, N\}$

$$\begin{aligned} & \partial_t \|d(\theta^0)\|_{L^2(\Omega)}^2 + \partial_t \|d(u_i^0)\|_{L^2(\Omega)}^2 + \partial_t \|d(v_i^0)\|_{L^2(\Omega)}^2 \\ & \leq C(M, \delta) \left(\|d(\theta^0)\|_{L^2(\Omega)}^2 + \|d(u_i^0)\|_{L^2(\Omega)}^2 + \|d(v_i^0)\|_{L^2(\Omega)}^2 + 1 \right). \end{aligned} \quad (2.37)$$

Hereby, we apply the Gronwall inequality to (2.37) and then integrate the resulting estimate over $(0, T)$ to obtain that

$$\|d(\theta^0)\|_{L^2((0, T) \times \Omega)}^2 + \|d(u_i^0)\|_{L^2((0, T) \times \Omega)}^2 + \|d(v_i^0)\|_{L^2((0, T) \times \Omega)}^2 \leq T^2 C(M, \delta) \exp(TC(M, \delta)). \quad (2.38)$$

Since $T^2 C(M, \delta) \exp(TC(M, \delta)) \rightarrow 0$ as $T \rightarrow 0$, we can construct an approximation scheme $(\theta^{0,n}, u_i^{0,n}, v_i^{0,n})$ for $n \in \mathbb{N}$ for the macro system in which the involved nonlinear terms are linearized. With a small enough T_0 such that $T_0^2 C(M, \delta) \exp(T_0 C(M, \delta)) < 1$, we claim that $\{\theta^{0,n}\}_{n \in \mathbb{N}}, \{u_i^{0,n}\}_{n \in \mathbb{N}}$ and $\{v_i^{0,n}\}_{n \in \mathbb{N}}$ are the Cauchy sequences in $K_1(M, T_0)$ by (2.38). Thus, the local existence and uniqueness of solutions in $[L^2((0, T) \times \Omega)]^{N+2}$ to (P^0) is guaranteed. \square

3 Main result

The main result of this paper is stated in the next Theorem whose applicability is delimited by the assumptions (A₁)-(A₂) and the extra regularity assumptions shall also be provided therein. Note that the involved macro reconstructions $\theta_0^\varepsilon, u_{i,0}^\varepsilon, v_{i,0}^\varepsilon$ for $i \in \{1, \dots, N\}$ shall be defined right in the next Subsection.

Theorem 12. *Assume (A₁)-(A₂). Let $(\theta^\varepsilon, u_i^\varepsilon, v_i^\varepsilon)$ and (θ^0, u_i^0, v_i^0) for $i \in \{1, \dots, N\}$ be weak solutions to (P^ε) and (P^0) in the sense of Definition 6 and Theorem 9, respectively. Let $\bar{\theta}, \bar{u}_i$ be the cell functions solving the cell problems (2.28)-(2.29) and satisfying*

$$\bar{\theta}, \bar{u}_i \in L^\infty(\Omega^\varepsilon; W_{\#}^{1+s,2}(Y_1)) \cap H^1(\Omega^\varepsilon; W_{\#}^{s,2}(Y_1)) \quad \text{for } s > d/2.$$

For every $t \in (0, T]$, we also assume that $\theta^0(t, \cdot), u_i^0(t, \cdot) \in W^{1,\infty}(\Omega^\varepsilon) \cap H^2(\Omega^\varepsilon)$ for $i \in \{1, \dots, N\}$. On top of that, we assume the initial homogenization limit is of the rate

$$\|\theta^{\varepsilon,0} - \theta^{0,0}\|_{L^2(\Omega^\varepsilon)}^2 + \sum_{i=1}^N \|u_i^{\varepsilon,0} - u_i^{0,0}\|_{L^2(\Omega^\varepsilon)}^2 + \sum_{i=1}^N \|v_i^{\varepsilon,0} - v_i^{0,0}\|_{L^2(\Gamma^\varepsilon)}^2 \leq \varepsilon^\gamma,$$

for some $\gamma \in \mathbb{R}_+$. Then the following corrector estimate holds

$$\begin{aligned} & \|\theta^\varepsilon - \theta^0\|_{L^2((0,T) \times \Omega^\varepsilon)}^2 + \sum_{i=1}^N \|u_i^\varepsilon - u_i^0\|_{L^2((0,T) \times \Omega^\varepsilon)}^2 \\ & + \|\nabla(\theta^\varepsilon - \theta^0)\|_{L^2(0,T;[L^2(\Omega^\varepsilon)]^d)}^2 + \sum_{i=1}^N \|\nabla(u_i^\varepsilon - u_i^0)\|_{L^2(0,T;[L^2(\Omega^\varepsilon)]^d)}^2 \leq C \max\{\varepsilon, \varepsilon^\gamma\}, \end{aligned}$$

where C is a generic positive constant that is independent of ε .

Furthermore, if $\gamma \geq 1$, then we obtain

$$\varepsilon \sum_{i=1}^N \|v_i^\varepsilon - v_i^0\|_{L^2((0,T) \times \Gamma^\varepsilon)}^2 \leq C\varepsilon.$$

3.1 Macroscopic reconstruction

To derive correctors estimates for our problem, we use the concept of the macroscopic reconstruction. We borrow this terminology from Eck[9], but note that it is also connected to similar concepts in the *a posteriori* numerical analysis of PDEs (see e.g. [26]). It turns out that we derive operators that could bring us the link between the strong formulations (P^ε) and (P^0) . For a.e. $t \in [0, T]$ and $x \in \Omega^\varepsilon$ we provide that

$$\theta_0^\varepsilon(t, x) := \theta^0(t, x), \quad (3.1)$$

$$u_{i,0}^\varepsilon(t, x) := u_i^0(t, x), \quad (3.2)$$

$$v_{i,0}^\varepsilon(t, x) := v_i^0(t, x). \quad (3.3)$$

Henceforward, we obtain the system of macroscopic reconstruction whose expression is similar to the strong formulations (P^0) , but acting on $x \in \Omega^\varepsilon$. We accordingly subtract this system from the microscopic system (P^ε) equation-by-equation and gain the difference system over Ω^ε . Then we proceed to the correctors justification by the following choice of test functions:

$$\varphi(t, x) := \theta^\varepsilon(t, x) - \left(\theta_0^\varepsilon(t, x) + \varepsilon m^\varepsilon(x) \bar{\theta}\left(x, \frac{x}{\varepsilon}\right) \cdot \nabla_x \theta^0(t, x) \right), \quad (3.4)$$

$$\phi_i(t, x) := u_i^\varepsilon(t, x) - \left(u_{i,0}^\varepsilon(t, x) + \varepsilon m^\varepsilon(x) \bar{u}_i\left(x, \frac{x}{\varepsilon}\right) \cdot \nabla_x u_i^0(t, x) \right), \quad (3.5)$$

where m_ε is a cut-off function with the properties (2.32).

Multiplying the difference system by the test functions $\varphi, \phi_i \in H^1(\Omega^\varepsilon)$ and integrating the resulting equations over Ω^ε , we obtain the system, denoted by $(\bar{\mathbb{P}}^\varepsilon)$, as follows:

$$\begin{aligned} & \int_{\Omega_0^\varepsilon} \partial_t (\theta^\varepsilon - \theta_0^\varepsilon) \varphi dx + \int_{\Omega^\varepsilon} (\kappa^\varepsilon \nabla \theta^\varepsilon - \mathbb{K} \nabla \theta_0^\varepsilon) \cdot \nabla \varphi dx + \varepsilon \int_{\Gamma_R^\varepsilon} g_0 \theta^\varepsilon \varphi dS_\varepsilon \\ & - g_0 \frac{|\Gamma_R|}{|Y_1|} \int_{\Omega^\varepsilon} \theta_0^\varepsilon \varphi dx = \int_{\Omega^\varepsilon} \left(\tau^\varepsilon \sum_{i=1}^N \nabla^\delta u_i^\varepsilon \cdot \nabla \theta^\varepsilon - \sum_{i=1}^N (\mathbb{T}^i \nabla^\delta u_{i,0}^\varepsilon) \cdot \nabla \theta_0^\varepsilon \right) \varphi dx, \\ & \int_{\Omega^\varepsilon} \partial_t (u_i^\varepsilon - u_{i,0}^\varepsilon) \phi_i dx + \int_{\Omega^\varepsilon} (d_i^\varepsilon \nabla u_i^\varepsilon - \mathbb{D}^i \nabla u_{i,0}^\varepsilon) \cdot \nabla \phi_i dx + \varepsilon \int_{\Gamma^\varepsilon} (a_i^\varepsilon u_i^\varepsilon - b_i^\varepsilon v_i^\varepsilon) \phi_i dS_\varepsilon \\ & - \int_{\Omega^\varepsilon} (A_i u_{i,0}^\varepsilon - B_i v_{i,0}^\varepsilon) \phi_i dx = \int_{\Omega^\varepsilon} (\rho_i^\varepsilon \nabla^\delta \theta^\varepsilon \cdot \nabla u_i^\varepsilon - (\mathbb{F}^i \nabla u_{i,0}^\varepsilon) \cdot \nabla^\delta \theta_0^\varepsilon) \phi_i dx \\ & + \int_{\Omega^\varepsilon} (R_i(u^\varepsilon) - R_i(u_0^\varepsilon)) \phi_i dx, \end{aligned}$$

According to the system $(\bar{\mathbb{P}}^\varepsilon)$, we denote the following terms:

$$\mathcal{I}_1 := \int_{\Omega^\varepsilon} \partial_t (\theta^\varepsilon - \theta_0^\varepsilon) \varphi dx, \quad (3.6)$$

$$\mathcal{I}_2 := \int_{\Omega^\varepsilon} (\kappa^\varepsilon \nabla \theta^\varepsilon - \mathbb{K} \nabla \theta_0^\varepsilon) \cdot \nabla \varphi dx, \quad (3.7)$$

$$\mathcal{I}_3 := \varepsilon \int_{\Gamma_R^\varepsilon} g_0 \theta^\varepsilon \varphi dS_\varepsilon - g_0 \frac{|\Gamma_R|}{|Y_1|} \int_{\Omega^\varepsilon} \theta_0^\varepsilon \varphi dx, \quad (3.8)$$

$$\mathcal{I}_4 := \int_{\Omega^\varepsilon} \left(\tau^\varepsilon \sum_{i=1}^N \nabla^\delta u_i^\varepsilon \cdot \nabla \theta^\varepsilon - \sum_{i=1}^N (\mathbb{T}^i \nabla^\delta u_i^0) \cdot \nabla \theta_0^\varepsilon \right) \varphi dx, \quad (3.9)$$

$$\mathcal{J}_1^i := \int_{\Omega^\varepsilon} \partial_t (u_i^\varepsilon - u_{i,0}^\varepsilon) \phi_i dx, \quad (3.10)$$

$$\mathcal{J}_2^i := \int_{\Omega^\varepsilon} (d_i^\varepsilon \nabla u_i^\varepsilon - \mathbb{D}^i \nabla u_{i,0}^\varepsilon) \cdot \nabla \phi_i dx, \quad (3.11)$$

$$\mathcal{J}_3^i := \varepsilon \int_{\Gamma^\varepsilon} (a_i^\varepsilon u_i^\varepsilon - b_i^\varepsilon v_i^\varepsilon) \phi_i dS_\varepsilon - \int_{\Omega^\varepsilon} (A_i u_{i,0}^\varepsilon - B_i v_{i,0}^\varepsilon) \phi_i dx, \quad (3.12)$$

$$\mathcal{J}_4^i := \int_{\Omega^\varepsilon} (\rho_i^\varepsilon \nabla^\delta \theta^\varepsilon \cdot \nabla u_i^\varepsilon - (\mathbb{F}^i \nabla u_{i,0}^\varepsilon) \cdot \nabla^\delta \theta_0^\varepsilon) \phi_i dx + \int_{\Omega^\varepsilon} (R_i(u^\varepsilon) - R_i(u_0^\varepsilon)) \phi_i dx. \quad (3.13)$$

We introduce, in the same spirit as for (3.1) and (3.2), another macroscopic reconstruction $\theta_1^\varepsilon(t, x)$ and $u_{i,1}^\varepsilon(t, x)$ defined as follows:

$$\theta_1^\varepsilon(t, x) := \theta_0^\varepsilon(t, x) + \varepsilon \bar{\theta} \left(x, \frac{x}{\varepsilon} \right) \cdot \nabla_x \theta^0(t, x),$$

$$u_{i,1}^\varepsilon(t, x) := u_{i,0}^\varepsilon(t, x) + \varepsilon \bar{u}_i \left(x, \frac{x}{\varepsilon} \right) \cdot \nabla_x u_i^0(t, x),$$

where $\bar{\theta}$ and \bar{u}_i are the cell functions introduced in Theorem 10.

By definition (3.1)-(3.2), the macroscopic reconstruction $\theta_0^\varepsilon(t, x)$ and $u_{i,0}^\varepsilon(t, x)$ are interchangeable, respectively, in notation with the limit functions $\theta^0(t, x)$ and $u_i^0(t, x)$ in Theorem 12.

3.2 Integral estimates

Remark 13. From Lemma 14, one can apply directly the L^2 -estimate between the space-dependent physical parameters of the microscopic problem (e.g. κ^ε , τ^ε) and their averages, even if the parameters in discussion are actually tensors. To this end, these estimates are controlled as $\|p^\varepsilon - \bar{p}\|_{L^2(\Omega^\varepsilon)} \leq C\varepsilon^{1/2}$, where p^ε refers to the oscillating coefficient and \bar{p} denotes its average.

Lemma 14. *Let Y_1 as defined in Subsection 2.1.1. Let $p^\varepsilon(x) := p(x/\varepsilon)$ belong to $H^1(\Omega^\varepsilon)$ satisfying*

$$\bar{p} := \frac{1}{|Y_1|} \int_{Y_1} p(y) dy.$$

Then the following estimate holds

$$\|p^\varepsilon - \bar{p}\|_{L^2(\Omega^\varepsilon)} \leq C\varepsilon^{1/2} \|p^\varepsilon\|_{H^1(\Omega^\varepsilon)}.$$

Proof. We consider the periodic geometry described in Figure 2.1 in Subsection 2.1.1. For a fixed test function $\phi \in H^1(\Omega^\varepsilon)$, we see that

$$\begin{aligned} \int_{\Omega^\varepsilon} (p^\varepsilon - \bar{p}) \phi dx &= \sum_{k \in \mathbb{Z}^d} \int_{\varepsilon Y_1^k} (p^\varepsilon - \bar{p}) \phi dx \\ &\leq C \int_{\varepsilon Y_1} (p^\varepsilon - \bar{p}) \phi dx. \end{aligned}$$

By changing the variable $x = \varepsilon y$, the relations

$$\begin{aligned} \int_{\varepsilon Y_1} p\left(\frac{x}{\varepsilon}\right) \phi(x) dx &= \varepsilon^d \int_{Y_1} p(y) \phi(\varepsilon y) dy, \\ \int_{\varepsilon Y_1} \int_{Y_1} p(y) \phi(x) dy dx &= \varepsilon^d \int_{Y_1} \int_{Y_1} p(y) \phi(\varepsilon z) dy dz, \end{aligned}$$

enable us to write:

$$\int_{\varepsilon Y_1} (p^\varepsilon - \bar{p}) \phi dx = \varepsilon^d |Y_1|^{-1} \int_{Y_1} \int_{Y_1} (p(y) \phi(\varepsilon y) - p(y) \phi(\varepsilon z)) dz dy. \quad (3.14)$$

Thanks to the representation

$$\phi(\varepsilon y) - \phi(\varepsilon z) = \varepsilon \int_0^1 \nabla \phi(t\varepsilon y + (1-t)\varepsilon z) \cdot (y - z) dt,$$

with $\xi = ty + (1-t)z$ and $\eta = y - z$, we note that (3.14) can be bounded from above by

$$\left| \int_{\varepsilon Y_1} (p^\varepsilon - \bar{p}) \phi dx \right| \leq \varepsilon^{d+1} |Y_1|^{-1} \left(\int_{Y_1} \int_{Y_2} |\nabla \phi(\varepsilon \xi) \cdot \eta|^2 d\eta d\xi \right)^{1/2} \left(\int_{Y_1} \int_{Y_1} |p(y)|^2 dy dz \right)^{1/2}. \quad (3.15)$$

In (3.15), we have denoted $Y_2 := \{y - z : \text{for } y, z \in Y_1\}$. Also, (3.15) leads to

$$\int_{\Omega^\varepsilon} (p^\varepsilon - \bar{p}) \phi dx \leq C\varepsilon \|p^\varepsilon\|_{L^2(\Omega^\varepsilon)} \|\nabla \phi\|_{L^2(\Omega^\varepsilon)},$$

and with $\phi = p^\varepsilon - \bar{p}$ and (2.33), (3.15) becomes $\|p^\varepsilon - \bar{p}\|_{L^2(\Omega^\varepsilon)}^2 \leq C\varepsilon \left(\|p^\varepsilon\|_{L^2(\Omega^\varepsilon)}^2 + \|\nabla p^\varepsilon\|_{L^2(\Omega^\varepsilon)}^2 \right)$ and hence, we finally get

$$\|p^\varepsilon - \bar{p}\|_{L^2(\Omega^\varepsilon)} \leq C\varepsilon^{1/2} \|p^\varepsilon\|_{H^1(\Omega^\varepsilon)}.$$

This completes the proof of the lemma. □

Due to the no-flux boundary condition (2.5), we define the function space

$$H^1(\Gamma_N^\varepsilon) := \{v \in H^1(\Gamma^\varepsilon) \mid -\kappa^\varepsilon \nabla v^\varepsilon \cdot \mathbf{n} = 0 \text{ on } \Gamma_N^\varepsilon\},$$

which is a closed subspace of $H^1(\Gamma^\varepsilon)$. This plays a role inside Lemma 15.

Lemma 15. *Let $\theta^\varepsilon \in L^2(0, T; H^1(\Gamma_N^\varepsilon))$ and $\theta^0 \in L^2(0, T; H^1(\Omega^\varepsilon))$. For any*

$$f_1 \in C([0, T]; H_+^1(\Omega^\varepsilon) \cap L_+^\infty(\Omega^\varepsilon)),$$

$$f_2 \in C([0, T]; H_+^1(\Gamma^\varepsilon) \cap L_+^\infty(\Gamma^\varepsilon)),$$

suppose that there exists $f_3 \in C[0, T]$ such that

$$\int_{\Omega^\varepsilon} f_1 \theta^0 dx = \int_{\Gamma_R^\varepsilon} f_2 \theta^\varepsilon dS_\varepsilon + \varepsilon f_3.$$

Then, it exists a $C > 0$ such that

$$\left| \int_{\Omega^\varepsilon} f_1 \theta^0 \varphi dx - \varepsilon \int_{\Gamma_R^\varepsilon} (f_2 \theta^\varepsilon + \varepsilon f_3) \varphi dS_\varepsilon \right| \leq \varepsilon C \|\varphi\|_{H^1(\Omega^\varepsilon)},$$

for any $\varphi \in H^1(\Omega^\varepsilon)$.

Proof. We adapt Lemma 5.2 from [28] to our context. The proof of the lemma is based on the following auxiliary problem: Given $f_1, f_2, \theta^\varepsilon, \theta^0$ as above and $\tilde{f} \in C[0, T]$, find Ψ such that

$$\begin{cases} \Delta_y \Psi(\cdot, x, y)|_{y=\frac{x}{\varepsilon}} = f_1 \theta^0 & \text{for } x \in \Omega^\varepsilon, \\ \nabla_y \Psi(\cdot, x, y) \cdot \mathbf{n} = f_2 \theta^\varepsilon + \varepsilon \tilde{f} & \text{for } (x, y) \in \Gamma_R^\varepsilon, \\ \nabla_y \Psi \cdot \mathbf{n} = 0 & \text{at } \Gamma_N^\varepsilon. \end{cases} \quad (3.16)$$

By [32, Lemma 2.1] and also [5], the problem (3.16) has a (weak) Y -periodic solution

$$\Psi(\cdot, x, y)|_{y=\frac{x}{\varepsilon}} \in L^2(0, T; H^1(\Omega^\varepsilon))$$

satisfying the integral equality

$$\int_{\Omega^\varepsilon} f_1 \theta^0 dx = \int_{\Gamma^\varepsilon} (f_2 \theta^\varepsilon + \varepsilon \tilde{f}) dS_\varepsilon = \int_{\Gamma_R^\varepsilon} f_2 \theta^\varepsilon dS_\varepsilon + \varepsilon f_3,$$

with f_3 being $|\Gamma_R^\varepsilon|^{-1} \tilde{f}$. Moreover, that solution is unique up to an additive constant.

Multiplying the first equation in (3.16) by $\varphi \in H^1(\Omega^\varepsilon)$ and then integrating the resulting equation over Ω^ε , we arrive at

$$\begin{aligned} \left| \int_{\Omega^\varepsilon} f_1 \theta^0 \varphi dx - \varepsilon \int_{\Gamma_R^\varepsilon} (f_2 \theta^\varepsilon + \varepsilon \tilde{f}) \varphi dS_\varepsilon \right| &= \left| \int_{\Omega^\varepsilon} \Delta_y \Psi(\cdot, x, y)|_{y=\frac{x}{\varepsilon}} \varphi dx - \right. \\ &\quad \left. - \varepsilon \int_{\Gamma_R^\varepsilon} f_2 \theta^\varepsilon \varphi dS_\varepsilon - \varepsilon^2 \int_{\Gamma_R^\varepsilon} \tilde{f} \varphi dS_\varepsilon \right| \end{aligned}$$

$$= \left| \int_{\Omega^\varepsilon} \varepsilon \left(\nabla_x \left[\nabla_y \Psi(\cdot, x, y) \Big|_{y=\frac{x}{\varepsilon}} \right] - \nabla_x \nabla_y \Psi(\cdot, x, y) \Big|_{y=\frac{x}{\varepsilon}} \right) \varphi - \right. \quad (3.17)$$

$$\left. - \varepsilon \int_{\Gamma_R^\varepsilon} f_2 \theta^\varepsilon \varphi dS_\varepsilon - \varepsilon^2 |\Gamma_R^\varepsilon|^{-1} \int_{\Gamma_R^\varepsilon} f_3 \varphi dS_\varepsilon \right| \quad (3.18)$$

$$= \left| \varepsilon \int_{\Gamma^\varepsilon} \left(\nabla_y \Psi(\cdot, x, y) \Big|_{y=\frac{x}{\varepsilon}} \cdot \mathbf{n} \varphi dS_\varepsilon - \varepsilon \int_{\Omega^\varepsilon} \nabla_y \Psi(\cdot, x, y) \Big|_{y=\frac{x}{\varepsilon}} \nabla_x \varphi dx \right) - \right. \quad (3.19)$$

$$\left. - \varepsilon \int_{\Omega^\varepsilon} \nabla_x \nabla_y \Psi(\cdot, x, y) \Big|_{y=\frac{x}{\varepsilon}} \varphi dx - \varepsilon \int_{\Gamma_R^\varepsilon} f_2 \theta^\varepsilon \varphi dS_\varepsilon - \varepsilon^2 |\Gamma_R^\varepsilon|^{-1} \int_{\Gamma_R^\varepsilon} f_3 \varphi dS_\varepsilon \right|. \quad (3.20)$$

Since $\Gamma^\varepsilon = \Gamma_R^\varepsilon \cup \Gamma_N^\varepsilon$, the choice of boundary conditions in (3.16) allows the boundary integrals in (3.20) to disappear. It follows from the triangle inequality and the Hölder inequality that

$$\begin{aligned} \left| \int_{\Omega^\varepsilon} f_1 \theta^0 \varphi dx - \varepsilon \int_{\Gamma_R^\varepsilon} \left(f_2 \theta^\varepsilon + \varepsilon \tilde{f} \right) \varphi dS_\varepsilon \right| &\leq \varepsilon \left(\left| \int_{\Omega^\varepsilon} \nabla_y \Psi(\cdot, x, y) \Big|_{y=\frac{x}{\varepsilon}} \nabla_x \varphi dx \right| + \right. \\ &\quad \left. + \left| \int_{\Omega^\varepsilon} \nabla_x \nabla_y \Psi(\cdot, x, y) \Big|_{y=\frac{x}{\varepsilon}} \varphi dx \right| \right) \\ &\leq C \varepsilon \|\varphi\|_{H^1(\Omega^\varepsilon)}. \end{aligned}$$

This completes the proof of the lemma. \square

4 Proof of Theorem 12

The proof of Theorem 12 relies on a fine control of the ε -dependence needed to estimate each term in (3.6)-(3.13). At first, the term \mathcal{I}_1 can be rewritten as:

$$\begin{aligned} \int_{\Omega^\varepsilon} \partial_t (\theta^\varepsilon - \theta^0) \left(\theta^\varepsilon - \theta^0 - \varepsilon m^\varepsilon \bar{\theta} \left(x, \frac{x}{\varepsilon} \right) \cdot \nabla_x \theta^0 \right) &= \frac{1}{2} \frac{d}{dt} \|\theta^\varepsilon(t) - \theta^0(t)\|_{L^2(\Omega^\varepsilon)}^2 \\ &\quad - \varepsilon \int_{\Omega^\varepsilon} \partial_t (\theta^\varepsilon - \theta^0) m^\varepsilon \bar{\theta} \left(x, \frac{x}{\varepsilon} \right) \cdot \nabla_x \theta^0 dx. \end{aligned} \quad (4.1)$$

Similarly, we proceed to estimate \mathcal{J}_1^i as follows:

$$\begin{aligned} \int_{\Omega^\varepsilon} \partial_t (u_i^\varepsilon - u_i^0) \left(u_i^\varepsilon - u_i^0 - \varepsilon m^\varepsilon \bar{u}_i \left(x, \frac{x}{\varepsilon} \right) \cdot \nabla_x u_i^0 \right) &= \frac{1}{2} \frac{d}{dt} \|u_i^\varepsilon(t) - u_i^0(t)\|_{L^2(\Omega^\varepsilon)}^2 \\ &\quad - \varepsilon \int_{\Omega^\varepsilon} \partial_t (u_i^\varepsilon - u_i^0) m^\varepsilon \bar{u}_i \left(x, \frac{x}{\varepsilon} \right) \cdot \nabla_x u_i^0 dx. \end{aligned} \quad (4.2)$$

Using the decomposition

$$\kappa^\varepsilon \nabla \theta^\varepsilon - \mathbb{K} \nabla \theta^0 = \kappa^\varepsilon \nabla (\theta^\varepsilon - \theta_1^\varepsilon) + \kappa^\varepsilon \nabla \theta_1^\varepsilon - \mathbb{K} \nabla \theta^0,$$

the term \mathcal{I}_2 thus becomes

$$\mathcal{I}_2 = \int_{\Omega^\varepsilon} \kappa^\varepsilon \nabla (\theta^\varepsilon - \theta_1^\varepsilon) \cdot \nabla \varphi dx + \int_{\Omega^\varepsilon} (\kappa^\varepsilon \nabla \theta_1^\varepsilon - \mathbb{K} \nabla \theta^0) \cdot \nabla \varphi dx. \quad (4.3)$$

Concerning the first term on the right-hand side of (4.3), we get

$$\int_{\Omega^\varepsilon} \kappa^\varepsilon \nabla (\theta^\varepsilon - \theta_1^\varepsilon) \cdot \nabla \varphi dx \geq \frac{\kappa_{\min}}{2} \|\nabla (\theta^\varepsilon - \theta_1^\varepsilon)(t)\|_{L^2(\Omega^\varepsilon)}^2 - C\varepsilon^2 \left\| \nabla ((1 - m^\varepsilon) \bar{\theta}^\varepsilon \cdot \nabla_x \theta^0(t)) \right\|_{L^2(\Omega^\varepsilon)}^2.$$

It is worth pointing out that the cell problems (2.28) and (2.29) require more regularity on the heat conductivity κ and the diffusion coefficient d_i , namely we need $\kappa, d_i \in H^1(\bar{Y}_1)$. On the other side, since these cell problems are elliptic problems on a non-convex polygon, it is well-known that the cell functions $\bar{\theta}$ and \bar{u}_i usually do not belong to $H^2(Y_1)$ in y no matter how smooth the right-hand sides of (2.28) and (2.29) are (cf. [19]). Due to the extra regularity on κ and d_i leading to their Lipschitz property in space and due to the Lipschitz boundary of the microstructure, the solutions can be at most in $H_{loc}^2(\bar{Y}_1)$ (see, e.g. [19, Theorem 2.2.2.3]). Notably, that result will not change even if the microstructure boundary is very smooth as in this case. We also emphasize that when investigating problems on domains without holes, the cell problems are then considered in the unit cell Y and by the convexity of that cell, one obtains the regularity of the cell functions up to $H^2(Y)$.

It follows from [35, Theorem 4] that the cell problems (2.28)-(2.29) admit a unique solution $(\bar{\theta}, \bar{u}_i) \in H_{\#}^{1+s}(Y_1) \times H_{\#}^{1+r}(Y_1)$ for some $s, r \in (-\frac{1}{2}, \frac{1}{2})$. Essentially, this hinders us when dealing with the term $\varepsilon \left\| \nabla ((1 - m^\varepsilon) \bar{\theta}^\varepsilon \cdot \nabla_x \theta^0(t)) \right\|_{L^2(\Omega^\varepsilon)}$. In fact, we need $\bar{\theta} \in L^\infty(\Omega^\varepsilon; C_{\#}^1(\bar{Y}_1))$, whereas its maximal regularity only gives $L^\infty(\Omega^\varepsilon; H_{\#}^{1+s}(Y_1))$ (a similar situation holds for \bar{u}_i). Recall the Sobolev embedding $W^{j+s,p}(Y_1) \subset C^j(\bar{Y}_1)$ for $sp > d$ (cf. [1]). Our Hilbertian framework, i.e. $p = 2, j = 1$, requires $s > d/2 \geq 1/2$ which leads to the impossibility of getting $C_{\#}^1(\bar{Y}_1)$ from $H_{\#}^{1+s}(Y_1)$. Obviously, one of the possibilities is to working with the domain without holes in 1D, i.e. $d = 1$ and $s = 1$. The fact that $(\bar{\theta}, \bar{u}_i) \in [L^\infty(\Omega^\varepsilon; W_{\#}^{1+s,2}(Y_1))]^2$ for $s > d/2$ is strictly needed to obtain $(\bar{\theta}, \bar{u}_i) \in [L^\infty(\Omega^\varepsilon; C_{\#}^1(\bar{Y}_1))]^2$. Then, with the assumption $\theta^0(t, \cdot) \in W^{1,\infty}(\Omega^\varepsilon) \cap H^2(\Omega^\varepsilon)$ and the extra regularity $\bar{\theta} \in H^1(\Omega^\varepsilon; W_{\#}^{s,2}(Y_1))$ providing $\bar{\theta} \in H^1(\Omega^\varepsilon; C_{\#}(\bar{Y}_1))$, we estimate that

$$\begin{aligned} \varepsilon \left\| \nabla ((1 - m^\varepsilon) \bar{\theta}^\varepsilon \cdot \nabla_x \theta^0(t)) \right\|_{L^2(\Omega^\varepsilon)} &\leq \varepsilon \|\nabla m^\varepsilon\|_{L^2(\Omega^\varepsilon)} \|\bar{\theta}\|_{L^\infty(\Omega^\varepsilon; C(\bar{Y}_1))} \|\theta^0(t)\|_{W^{1,\infty}(\Omega^\varepsilon)} \\ &\quad + \varepsilon \|\nabla_x \bar{\theta}\|_{L^2(\Omega^\varepsilon; C(\bar{Y}_1))} \|\theta^0(t)\|_{W^{1,\infty}(\Omega^\varepsilon)} \\ &\quad + \|1 - m^\varepsilon\|_{L^2(\Omega^\varepsilon)} \|\nabla_y \bar{\theta}\|_{L^\infty(\Omega^\varepsilon; C(\bar{Y}_1))} \|\theta^0(t)\|_{W^{1,\infty}(\Omega^\varepsilon)} \\ &\quad + \varepsilon \|\bar{\theta}\|_{L^\infty(\Omega^\varepsilon; C(\bar{Y}_1))} \|\theta^0(t)\|_{H^2(\Omega^\varepsilon)} \\ &\leq C (\varepsilon + \varepsilon^{1/2}), \end{aligned}$$

where we use the inequalities (2.32) together with the fact that $\nabla = \nabla_x + \varepsilon^{-1} \nabla_y$.

Observe that

$$\nabla \theta_1^\varepsilon = \nabla_x \theta^0 + (\nabla_y \bar{\theta})^\varepsilon \nabla_x \theta^0 + \varepsilon \bar{\theta}^\varepsilon \nabla_x \nabla \theta^0 + \varepsilon (\nabla_x \bar{\theta})^\varepsilon \nabla_x \theta^0. \quad (4.4)$$

Hence, we get

$$\begin{aligned} \kappa^\varepsilon \nabla \theta_1^\varepsilon - \mathbb{K} \nabla \theta^0 &= \kappa^\varepsilon (\nabla \theta^0 + (\nabla_y \bar{\theta})^\varepsilon \nabla_x \theta^0) - \mathbb{K} \nabla \theta^0 \\ &\quad + \kappa^\varepsilon \varepsilon (\bar{\theta}^\varepsilon \nabla_x \nabla \theta^0 + (\nabla_x \bar{\theta})^\varepsilon \nabla_x \theta^0). \end{aligned} \quad (4.5)$$

We note that the L^2 -norm of the second term on the right-hand side of (4.5) is bounded from above by

$$\begin{aligned} \varepsilon \left\| \kappa^\varepsilon (\bar{\theta}^\varepsilon \nabla_x \nabla \theta^0 + (\nabla_x \bar{\theta})^\varepsilon \nabla_x \theta^0) \right\|_{L^2(\Omega^\varepsilon)} &\leq C\varepsilon \|\bar{\theta}\|_{L^\infty(\Omega^\varepsilon; C(\bar{Y}_1))} \|\theta^0\|_{H^2(\Omega^\varepsilon)} \\ &\quad + C\varepsilon \|\nabla_x \bar{\theta}\|_{L^2(\Omega^\varepsilon; C(\bar{Y}_1))} \|\theta^0\|_{W^{1,\infty}(\Omega^\varepsilon)}. \end{aligned}$$

Let us handle now the remaining quantity $\kappa^\varepsilon (\nabla \theta^0 + (\nabla_y \bar{\theta})^\varepsilon \nabla_x \theta^0) - \mathbb{K} \nabla \theta^0$. In fact, recall that $\mathcal{G} := \kappa(\mathbb{I} + \nabla_y \bar{\theta}) - \mathbb{K}$ is divergence-free with respect to $y \in Y_1$ due to the structure of the cell problems in Theorem 10. Moreover, we know that its average also vanishes, i.e.

$$\int_{Y_1} \mathcal{G} dy = 0,$$

by virtue of the definition of the homogenized heat conductivity \mathbb{K} in Theorem 8.

As a consequence, \mathcal{G} possesses a vector potential \mathbf{V} and this vector potential is skew-symmetric such that $\mathcal{G} = \nabla_y \mathbf{V}$. In general, the selection of the vector potential is non-unique. However, we can choose \mathbf{V} to solve the Poisson equation $\Delta_y \mathbf{V} = \eta(x, y) \nabla_y \mathcal{G}$ for some function η just depending on the dimensions. Using this equation together with the periodic boundary conditions at ∂Y_0 and the vanishing cell average, we can determine this vector potential \mathbf{V} uniquely. Now, we formulate the quantity $\mathcal{G}^\varepsilon \nabla \theta^0 = \kappa^\varepsilon (\nabla \theta^0 + (\nabla_y \bar{\theta})^\varepsilon \nabla_x \theta^0) - \mathbb{K} \nabla \theta^0$ in terms of this vector potential. Using the relation that $\nabla_y = \varepsilon \nabla - \varepsilon \nabla_x$, we have

$$\mathcal{G}^\varepsilon \nabla \theta^0 = \varepsilon \nabla \cdot (\mathbf{V}^\varepsilon \nabla \theta^0) - \varepsilon \mathbf{V}^\varepsilon \Delta \theta^0 - \varepsilon (\nabla_x \mathbf{V})^\varepsilon \nabla \theta^0. \quad (4.6)$$

Due to the skew-symmetry of \mathbf{V} (and also that of \mathbf{V}^ε), the first term on the right-hand side of (4.6) is divergence-free, indicating the boundedness in $L^2(\Omega^\varepsilon)$ with the order of $\mathcal{O}(\varepsilon)$. In addition, combining $\bar{\theta} \in L^\infty(\Omega^\varepsilon; W_{\#}^{1+s,2}(Y_1)) \cap H^1(\Omega^\varepsilon; W_{\#}^{s,2}(Y_1))$ with the above Poisson equation $\Delta_y \mathbf{V} = \eta(x, y) \nabla_y \mathcal{G}$ yields

$$\|\mathbf{V}\|_{W^{1+s,2}(Y_1)} \leq C \|\mathcal{G}\|_{W^{s,2}(Y_1)}.$$

By the compact embedding $W^{s,2}(Y_1) \subset C(\bar{Y}_1)$ for $s > d/2 \geq 1$, we thus get

$$\mathbf{V} \in L^\infty(\Omega^\varepsilon; C_{\#}(\bar{Y}_1)) \cap H^1(\Omega^\varepsilon; C_{\#}(\bar{Y}_1)).$$

As a consequence, the boundedness in $L^2(\Omega^\varepsilon)$ of the second and third terms on the right-hand side of (4.6) is given by

$$\varepsilon \|\mathbf{V}^\varepsilon \Delta \theta^0 + (\nabla_x \mathbf{V})^\varepsilon \nabla \theta^0\|_{L^2(\Omega^\varepsilon)} \leq \varepsilon \|\mathbf{V}\|_{L^\infty(\Omega^\varepsilon; C(\bar{Y}_1))} \|\theta^0\|_{H^2(\Omega^\varepsilon)} + \varepsilon \|\mathbf{V}\|_{H^1(\Omega^\varepsilon; C(\bar{Y}_1))} \|\theta^0\|_{W^{1,\infty}(\Omega^\varepsilon)}.$$

Therefore, with the help of the Hölder inequality, we note that

$$\int_{\Omega^\varepsilon} (\kappa^\varepsilon \nabla \theta_1^\varepsilon - \mathbb{K} \nabla \theta^0) \cdot \nabla \varphi dx \leq C \varepsilon,$$

which completes the estimates for \mathcal{I}_2 .

Consequently, we can write

$$\mathcal{I}_2 \geq C \|\nabla (\theta^\varepsilon - \theta_1^\varepsilon)(t)\|_{[L^2(\Omega^\varepsilon)]^d}^2 - C(\varepsilon^2 + \varepsilon). \quad (4.7)$$

Similarly, estimating the term \mathcal{J}_2^i leads to

$$\mathcal{J}_2^i \geq C \|\nabla (u_i^\varepsilon - u_{i,1}^\varepsilon)(t)\|_{[L^2(\Omega^\varepsilon)]^d}^2 - C(\varepsilon^2 + \varepsilon). \quad (4.8)$$

Concerning the estimate of the term \mathcal{I}_3 , we note the following: Thanks to the compatibility constraint (Theorem 15) with the choice $\varphi = \theta^\varepsilon - \theta^0$, we get that

$$\begin{aligned} \mathcal{I}_3 &\leq C \varepsilon \|\varphi\|_{H^1(\Omega^\varepsilon)} \\ &\leq C \varepsilon \left(\|\theta^\varepsilon - \theta^0\|_{L^2(\Omega^\varepsilon)} + \|\nabla (\theta^\varepsilon - \theta_1^\varepsilon)\|_{[L^2(\Omega^\varepsilon)]^d} + \|\nabla (\theta_1^\varepsilon - \theta^0)\|_{[L^2(\Omega^\varepsilon)]^d} \right) \\ &\leq C \varepsilon \left(\|\theta^\varepsilon - \theta^0\|_{L^2(\Omega^\varepsilon)} + \|\nabla (\theta^\varepsilon - \theta_1^\varepsilon)\|_{[L^2(\Omega^\varepsilon)]^d} + C(1 + \varepsilon) \right), \end{aligned} \quad (4.9)$$

where we use again the difference relation (4.4) and get the following bound from above

$$\begin{aligned} \|\nabla(\theta_1^\varepsilon - \theta^0)\|_{L^2(\Omega^\varepsilon)} &\leq \|\nabla_y \bar{\theta}\|_{L^\infty(\Omega^\varepsilon; C(\bar{Y}_1))} \|\theta^0\|_{W^{1,\infty}(\Omega^\varepsilon)} \\ &\quad + \varepsilon \left(\|\bar{\theta}\|_{L^\infty(\Omega^\varepsilon; C(\bar{Y}_1))} \|\theta^0\|_{H^2(\Omega^\varepsilon)} + \|\nabla_x \bar{\theta}\|_{L^2(\Omega^\varepsilon; C(\bar{Y}_1))} \|\theta^0\|_{W^{1,\infty}(\Omega^\varepsilon)} \right). \end{aligned}$$

Similarly, the term \mathcal{J}_3^i is bounded from above by

$$\mathcal{J}_3^i \leq C\varepsilon \left(\|u_i^\varepsilon - u_i^0\|_{L^2(\Omega^\varepsilon)} + \|\nabla(u_i^\varepsilon - u_{i,1}^\varepsilon)\|_{[L^2(\Omega^\varepsilon)]^d} + C(1 + \varepsilon) \right). \quad (4.10)$$

Note the elementary decomposition:

$$\begin{aligned} \tau^\varepsilon \nabla^\delta u_i^\varepsilon \cdot \nabla \theta^\varepsilon - (\mathbb{T}^i \nabla^\delta u_i^0) \cdot \nabla \theta^0 &= (\tau^\varepsilon - \mathbb{T}^i) \nabla^\delta u_i^\varepsilon \cdot \nabla \theta^\varepsilon \\ &\quad + \mathbb{T}^i (\nabla^\delta u_i^\varepsilon - \nabla^\delta u_i^0) \cdot \nabla \theta^\varepsilon + \mathbb{T}^i (\nabla \theta^\varepsilon - \nabla \theta^0) \cdot \nabla^\delta u_i^0. \end{aligned}$$

Multiplying the above equation by the test function φ , we arrive at

$$\begin{aligned} (\tau^\varepsilon \nabla^\delta u_i^\varepsilon \cdot \nabla \theta^\varepsilon - (\mathbb{T}^i \nabla^\delta u_i^0) \cdot \nabla \theta^0) \varphi &= (\tau^\varepsilon - \mathbb{T}^i) \nabla^\delta u_i^\varepsilon \cdot \nabla \theta^\varepsilon (\theta^\varepsilon - \theta^0) \\ &\quad - \varepsilon (\tau^\varepsilon - \mathbb{T}^i) \nabla^\delta u_i^\varepsilon \cdot \nabla \theta^\varepsilon m^\varepsilon \bar{\theta}^\varepsilon \cdot \nabla_x \theta^0 \\ &\quad + \mathbb{T}^i (\nabla^\delta u_i^\varepsilon - \nabla^\delta u_i^0) \cdot \nabla \theta^\varepsilon (\theta^\varepsilon - \theta^0) \\ &\quad - \varepsilon \mathbb{T}^i (\nabla^\delta u_i^\varepsilon - \nabla^\delta u_i^0) \cdot \nabla \theta^\varepsilon m^\varepsilon \bar{\theta}^\varepsilon \cdot \nabla_x \theta^0 \\ &\quad + \mathbb{T}^i (\nabla \theta^\varepsilon - \nabla \theta^0) \cdot \nabla^\delta u_i^0 (\theta^\varepsilon - \theta^0) \\ &\quad - \varepsilon \mathbb{T}^i (\nabla \theta^\varepsilon - \nabla \theta^0) \cdot \nabla^\delta u_i^0 m^\varepsilon \bar{\theta}^\varepsilon \cdot \nabla_x \theta^0 \\ &= \sum_{k=1}^6 \mathcal{I}_4^k. \end{aligned}$$

To be able to estimate \mathcal{I}_4 , we need to ensure the boundedness of each of the terms $\int_{\Omega^\varepsilon} \mathcal{I}_4^{k_i}$ for $k_i \in \{1, \dots, 6\}$ and $i \in \{1, \dots, N\}$. We obtain:

$$\begin{aligned} \int_{\Omega^\varepsilon} |\mathcal{I}_4^2| dx &\leq \varepsilon \|\nabla^\delta u_i^\varepsilon \cdot \nabla \theta^\varepsilon\|_{L^2(\Omega^\varepsilon)} \left\| (\tau^\varepsilon - \mathbb{T}^i) m^\varepsilon \bar{\theta} \left(\frac{x}{\varepsilon} \right) \cdot \nabla_x \theta^0 \right\|_{L^2(\Omega^\varepsilon)} \\ &\leq \varepsilon \|u_i^\varepsilon\|_{L^\infty(\Omega^\varepsilon)} \|\nabla \theta^\varepsilon\|_{[L^2(\Omega^\varepsilon)]^d} \|\bar{\theta}\|_{L^\infty(\Omega^\varepsilon; C(Y_1))} \|\theta^0\|_{W^{1,\infty}(\Omega^\varepsilon)} \|\tau^\varepsilon - \mathbb{T}^i\|_{L^2(\Omega^\varepsilon)}, \quad (4.11) \end{aligned}$$

and

$$\begin{aligned} \int_{\Omega^\varepsilon} |\mathcal{I}_4^4| dx &\leq \frac{\varepsilon}{2} |\mathbb{T}^i| \left\| (\nabla^\delta u_i^\varepsilon - \nabla^\delta u_i^0) \cdot \nabla \theta^\varepsilon \right\|_{L^2(\Omega^\varepsilon)} \left\| m^\varepsilon \bar{\theta} \left(\frac{x}{\varepsilon} \right) \cdot \nabla_x \theta^0 \right\|_{L^2(\Omega^\varepsilon)} \\ &\leq \frac{\varepsilon}{2} |\mathbb{T}^i| C_\delta^2 \|u_i^\varepsilon - u_i^0\|_{L^2(\Omega^\varepsilon)} \|\nabla \theta^\varepsilon\|_{[L^2(\Omega^\varepsilon)]^d} \|\bar{\theta}\|_{L^\infty(\Omega^\varepsilon; C(Y_1))} \|\theta^0\|_{W^{1,\infty}(\Omega^\varepsilon)}. \quad (4.12) \end{aligned}$$

Furthermore, we estimate

$$\begin{aligned} \int_{\Omega^\varepsilon} |\mathcal{I}_4^1| dx &\leq \|\tau^\varepsilon - \mathbb{T}^i\|_{L^2(\Omega^\varepsilon)} \|\nabla^\delta u_i^\varepsilon \cdot \nabla \theta^\varepsilon\|_{L^2(\Omega^\varepsilon)} \|\theta^\varepsilon - \theta^0\|_{L^\infty(\Omega^\varepsilon)} \\ &\leq C_\delta \|\tau^\varepsilon - \mathbb{T}^i\|_{L^2(\Omega^\varepsilon)} \|u_i^\varepsilon\|_{L^\infty(\Omega^\varepsilon)} \|\nabla \theta^\varepsilon\|_{[L^2(\Omega^\varepsilon)]^d} \left(\|\theta^\varepsilon\|_{L^\infty(\Omega^\varepsilon)} + \|\theta^0\|_{W^{1,\infty}(\Omega^\varepsilon)} \right), \quad (4.13) \end{aligned}$$

and by Young's inequality, it yields

$$\begin{aligned} \int_{\Omega^\varepsilon} |\mathcal{I}_4^3| dx &\leq \frac{|\mathbb{T}^i|^2}{2} C_\delta^2 \|\nabla^\delta u_i^\varepsilon - \nabla^\delta u_i^0\|_{L^\infty(\Omega^\varepsilon)}^2 \|\nabla \theta^\varepsilon\|_{[L^2(\Omega^\varepsilon)]^d}^2 + \frac{1}{2} \|\theta^\varepsilon - \theta^0\|_{L^2(\Omega^\varepsilon)}^2 \\ &\leq \frac{|\mathbb{T}^i|^2}{2} C_\delta^4 \|\nabla \theta^\varepsilon\|_{[L^2(\Omega^\varepsilon)]^d}^2 \|u_i^\varepsilon - u_i^0\|_{L^2(\Omega^\varepsilon)}^2 + \frac{1}{2} \|\theta^\varepsilon - \theta^0\|_{L^2(\Omega^\varepsilon)}^2, \quad (4.14) \end{aligned}$$

and

$$\begin{aligned} \int_{\Omega^\varepsilon} |\mathcal{I}_4^5| dx &\leq \frac{|\mathbb{T}_i|^2}{2} \|(\nabla \theta^\varepsilon - \nabla \theta^0) \cdot \nabla^\delta u_i^0\|_{L^2(\Omega^\varepsilon)}^2 + \frac{1}{2} \|\theta^\varepsilon - \theta^0\|_{L^2(\Omega^\varepsilon)}^2 \\ &\leq \frac{|\mathbb{T}_i|^2}{2} C_\delta^2 \|u_i^0\|_{L^\infty(\Omega^\varepsilon)}^2 \|\nabla \theta^\varepsilon - \nabla \theta^0\|_{[L^2(\Omega^\varepsilon)]^d}^2 + \frac{1}{2} \|\theta^\varepsilon - \theta^0\|_{L^2(\Omega^\varepsilon)}^2, \end{aligned} \quad (4.15)$$

$$\int_{\Omega^\varepsilon} |\mathcal{I}_4^6| dx \leq \frac{\varepsilon |\mathbb{T}_i|^2}{2} \|u_i^0\|_{L^\infty(\Omega^\varepsilon)}^2 \|\nabla \theta^\varepsilon - \nabla \theta^0\|_{[L^2(\Omega^\varepsilon)]^d}^2 + \frac{\varepsilon}{2} \|\bar{\theta}\|_{L^\infty(\Omega^\varepsilon; C(\bar{Y}_1))}^2 \|\theta^0\|_{W^{1,\infty}(\Omega^\varepsilon)}^2. \quad (4.16)$$

Remark that the first integral in \mathcal{J}_4^i can be estimated similarly. On top of that, observe that we can find constants $C_{R_i} > 0$ (independent of ε) such that

$$\|R_i(u^\varepsilon) - R_i(u^0)\|_{L^2(\Omega^\varepsilon)} \leq C_{R_i} \sum_{j=1}^N \|u_j^\varepsilon - u_j^0\|_{L^2(\Omega^\varepsilon)} \quad \text{for } i \in \{1, \dots, N\},$$

in which the constants C_{R_i} depend on the L^∞ -bounds of the concentrations u^ε, u^0 as discussed in [22, Section 5].

The estimate on the second integral of \mathcal{J}_4^i can be computed directly. Note that for $i \in \{1, \dots, N\}$, we have:

$$\begin{aligned} (R_i(u^\varepsilon) - R_i(u^0)) \phi_i &= (R_i(u^\varepsilon) - R_i(u^0)) (u_i^\varepsilon - u_i^0) \\ &\quad - \varepsilon (R_i(u^\varepsilon) - R_i(u^0)) m^\varepsilon \bar{u}_i \left(\frac{x}{\varepsilon} \right) \cdot \nabla_x u_i^0. \end{aligned}$$

This gives

$$\begin{aligned} \int_{\Omega^\varepsilon} (R_i(u^\varepsilon) - R_i(u^0)) \phi_i dx &\leq C_{R_i} \sum_{j=1}^N \|u_j^\varepsilon - u_j^0\|_{L^2(\Omega^\varepsilon)} \left(\|u_i^\varepsilon - u_i^0\|_{L^2(\Omega^\varepsilon)} + \right. \\ &\quad \left. + \varepsilon \|\bar{u}_i\|_{L^\infty(\Omega^\varepsilon; C(\bar{Y}_1))} \|u_i^0\|_{W^{1,\infty}(\Omega^\varepsilon)} \right). \end{aligned} \quad (4.17)$$

Collecting the estimates (4.7), (4.8), (4.9), (4.10), (4.11)-(4.16) and (4.17), we obtain:

$$\begin{aligned} &\|\nabla(\theta^\varepsilon - \theta^0)(t)\|_{[L^2(\Omega^\varepsilon)]^d}^2 + \sum_{i=1}^N \|\nabla(u_i^\varepsilon - u_i^0)(t)\|_{[L^2(\Omega^\varepsilon)]^d}^2 \\ &\leq C(\varepsilon^2 + \varepsilon) + C\varepsilon \left(\|\theta^\varepsilon(t) - \theta^0(t)\|_{L^2(\Omega^\varepsilon)} + \|\nabla(\theta^\varepsilon - \theta^0)(t)\|_{[L^2(\Omega^\varepsilon)]^d} + C(1 + \varepsilon) \right) \\ &\quad + C\varepsilon \sum_{i=1}^N \left(\|u_i^\varepsilon(t) - u_i^0(t)\|_{L^2(\Omega^\varepsilon)} + \|\nabla(u_i^\varepsilon - u_i^0)(t)\|_{[L^2(\Omega^\varepsilon)]^d} + C(1 + \varepsilon) \right) \\ &\quad + C \left(\|\tau^\varepsilon - \mathbb{T}^i\|_{L^2(\Omega^\varepsilon)} \|\theta^\varepsilon(t) - \theta^0(t)\|_{L^2(\Omega^\varepsilon)} + \sum_{i=1}^N \|\rho_i^\varepsilon - \mathbb{F}^i\|_{L^2(\Omega^\varepsilon)} \|u_i^\varepsilon(t) - u_i^0(t)\|_{L^2(\Omega^\varepsilon)} \right) \\ &\quad + C\varepsilon \left(\sum_{i=1}^N \|u_i^\varepsilon(t) - u_i^0(t)\|_{L^2(\Omega^\varepsilon)} + \|\theta^\varepsilon(t) - \theta^0(t)\|_{L^2(\Omega^\varepsilon)} \right) \\ &\quad + C \left(\sum_{i=1}^N \|u_i^\varepsilon(t) - u_i^0(t)\|_{L^2(\Omega^\varepsilon)}^2 + \|\theta^\varepsilon(t) - \theta^0(t)\|_{L^2(\Omega^\varepsilon)}^2 \right) \\ &\quad + C\varepsilon \left(\|\nabla(\theta^\varepsilon - \theta^0)(t)\|_{[L^2(\Omega^\varepsilon)]^d}^2 + \sum_{i=1}^N \|\nabla(u_i^\varepsilon - u_i^0)(t)\|_{[L^2(\Omega^\varepsilon)]^d}^2 \right) + C\varepsilon. \end{aligned}$$

Notably, Theorem 14 provides us that the L^2 -error estimates between the Soret and Dufour coefficients and their homogenized (averaged) versions, i.e. $\|\tau^\varepsilon - \mathbb{T}^i\|_{L^2(\Omega^\varepsilon)}$ and $\|\rho_i^\varepsilon - \mathbb{F}^i\|_{L^2(\Omega^\varepsilon)}$ are of the order $\mathcal{O}(\varepsilon^{1/2})$. It thus yields that

$$\begin{aligned}
& \|\nabla(\theta^\varepsilon - \theta_1^\varepsilon)(t)\|_{[L^2(\Omega^\varepsilon)]^d}^2 + \sum_{i=1}^N \|\nabla(u_i^\varepsilon - u_{i,1}^\varepsilon)(t)\|_{[L^2(\Omega^\varepsilon)]^d}^2 \\
& \leq C(\varepsilon^2 + \varepsilon) + C\varepsilon \left(\|\theta^\varepsilon(t) - \theta^0(t)\|_{L^2(\Omega^\varepsilon)} + \|\nabla(\theta^\varepsilon - \theta_1^\varepsilon)(t)\|_{[L^2(\Omega^\varepsilon)]^d} \right) \\
& \quad + C\varepsilon \sum_{i=1}^N \left(\|u_i^\varepsilon(t) - u_i^0(t)\|_{L^2(\Omega^\varepsilon)} + \|\nabla(u_i^\varepsilon - u_{i,1}^\varepsilon)(t)\|_{[L^2(\Omega^\varepsilon)]^d} \right) \\
& \quad + C\varepsilon^{1/2} \left(\|\theta^\varepsilon(t) - \theta^0(t)\|_{L^2(\Omega^\varepsilon)} + \sum_{i=1}^N \|u_i^\varepsilon(t) - u_i^0(t)\|_{L^2(\Omega^\varepsilon)} \right) \\
& \quad + C \left(\sum_{i=1}^N \|u_i^\varepsilon(t) - u_i^0(t)\|_{L^2(\Omega^\varepsilon)}^2 + \|\theta^\varepsilon(t) - \theta^0(t)\|_{L^2(\Omega^\varepsilon)}^2 \right) \\
& \quad + C\varepsilon \left(\|\nabla(\theta^\varepsilon - \theta_1^\varepsilon)(t)\|_{[L^2(\Omega^\varepsilon)]^d}^2 + \sum_{i=1}^N \|\nabla(u_i^\varepsilon - u_{i,1}^\varepsilon)(t)\|_{[L^2(\Omega^\varepsilon)]^d}^2 \right). \tag{4.18}
\end{aligned}$$

It now remains to estimate the second term on the right-hand side of (4.1)-(4.2). In fact, integrating by parts gives

$$\begin{aligned}
\int_0^t \int_{\Omega^\varepsilon} m^\varepsilon \partial_t (u_i^\varepsilon - u_i^0) \bar{u}_i \left(\frac{x}{\varepsilon} \right) \cdot \nabla_x u_i^0(s, x) dx ds &= \int_{\Omega^\varepsilon} m^\varepsilon (u_i^\varepsilon - u_i^0) \bar{u}_i \left(\frac{x}{\varepsilon} \right) \cdot \nabla_x u_i^0(s, x) dx \Big|_{s=0}^{s=t} \\
&\quad - \int_0^t \int_{\Omega^\varepsilon} m^\varepsilon (u_i^\varepsilon - u_i^0) \bar{u}_i \left(\frac{x}{\varepsilon} \right) \cdot \nabla_x \partial_t u_i^0(s, x) dx ds.
\end{aligned}$$

We then observe that

$$\begin{aligned}
& \varepsilon \left| \int_{\Omega^\varepsilon} m^\varepsilon [(u_i^\varepsilon - u_i^0) - (u_i^\varepsilon(0) - u_i^0(0))] \bar{u}_i^\varepsilon \cdot \nabla_x u_i^0(t, x) dx \right| \\
& \leq C\varepsilon \left(\|u_i^\varepsilon(t) - u_i^0(t)\|_{L^2(\Omega^\varepsilon)} + \|u_i^{\varepsilon,0} - u_i^{0,0}\|_{L^2(\Omega^\varepsilon)} \right),
\end{aligned}$$

and hence,

$$\begin{aligned}
& \varepsilon \left| \int_{\Omega^\varepsilon} m^\varepsilon [(\theta^\varepsilon - \theta^0) - (\theta^\varepsilon(0) - \theta^0(0))] \bar{\theta}^\varepsilon \cdot \nabla_x \theta^0(t, x) dx \right| \\
& \leq C\varepsilon \left(\|\theta^\varepsilon(t) - \theta^0(t)\|_{L^2(\Omega^\varepsilon)} + \|\theta^{\varepsilon,0} - \theta^{0,0}\|_{L^2(\Omega^\varepsilon)} \right).
\end{aligned}$$

For all $t \in (0, T]$, we set

$$\begin{aligned}
w_1(t) &= \|\theta^\varepsilon(t) - \theta^0(t)\|_{L^2(\Omega^\varepsilon)}^2 + \sum_{i=1}^N \|u_i^\varepsilon(t) - u_i^0(t)\|_{L^2(\Omega^\varepsilon)}^2, \\
w_2(t) &= \|\nabla(\theta^\varepsilon - \theta_1^\varepsilon)(t)\|_{[L^2(\Omega^\varepsilon)]^d}^2 + \sum_{i=1}^N \|\nabla(u_i^\varepsilon - u_{i,1}^\varepsilon)(t)\|_{[L^2(\Omega^\varepsilon)]^d}^2, \\
w_0 &= \|\theta^{\varepsilon,0} - \theta^{0,0}\|_{L^2(\Omega^\varepsilon)}^2 + \sum_{i=1}^N \|u_i^{\varepsilon,0} - u_i^{0,0}\|_{L^2(\Omega^\varepsilon)}^2.
\end{aligned}$$

Then, when integrating (4.18) and (4.1)-(4.2) from 0 to t , we are led to the following Gronwall-like estimate

$$w_1(t) + \int_0^t w_2(s) ds \leq C \left(\varepsilon^2 + \varepsilon + (1 + \varepsilon) w_0 + \varepsilon \int_0^t w_1(s) ds \right),$$

which can be rewritten as

$$w_1(t) + \int_0^t w_2(s) ds \leq C(\varepsilon + (1 + \varepsilon) w_0) e^{C\varepsilon t} \quad \text{for } t \in [0, T]. \quad (4.19)$$

Finally, we turn our attention to the corrector estimate for v_i^ε . For $i \in \{1, \dots, N\}$ we consider the equation for the reconstruction $v_{i,0}^\varepsilon = v_i^0$, obtained from (2.18), with the test function $\psi_i \in L^2(\Gamma^\varepsilon)$ and integrate the resulting equation over Γ^ε to get

$$\varepsilon \int_{\Gamma^\varepsilon} \partial_t v_i^0 \psi_i dS_\varepsilon = \varepsilon \int_{\Gamma^\varepsilon} (A_i u_i^0 - B_i v_i^0) \psi_i dS_\varepsilon. \quad (4.20)$$

Then, we find the difference equation for the micro concentration v_i^ε and the reconstruction v_i^0 by subtracting the third equation of (2.13) and (4.20), provided that

$$\begin{aligned} \varepsilon \int_{\Gamma^\varepsilon} \partial_t (v_i^\varepsilon - v_i^0) \psi_i dS_\varepsilon &= \varepsilon \int_{\Gamma^\varepsilon} (a_i^\varepsilon u_i^\varepsilon - A_i u_i^0) \psi_i dS_\varepsilon - \varepsilon \int_{\Gamma^\varepsilon} (b_i^\varepsilon v_i^\varepsilon - B_i v_i^0) \psi_i dS_\varepsilon \\ &= \varepsilon \int_{\Gamma^\varepsilon} [a_i^\varepsilon (u_i^\varepsilon - u_i^0) + (a_i^\varepsilon - A_i) u_i^0] \psi_i dS_\varepsilon \\ &\quad - \varepsilon \int_{\Gamma^\varepsilon} [b_i^\varepsilon (v_i^\varepsilon - v_i^0) + (b_i^\varepsilon - B_i) v_i^0] \psi_i dS_\varepsilon. \end{aligned}$$

Hereby, we choose $\psi_i = v_i^\varepsilon - v_i^0$ to obtain the following estimate

$$\begin{aligned} \frac{\varepsilon}{2} \frac{d}{dt} \|v_i^\varepsilon - v_i^0\|_{L^2(\Gamma^\varepsilon)}^2 &\leq C\varepsilon \left(\|u_i^\varepsilon - u_i^0\|_{L^2(\Gamma^\varepsilon)}^2 + \|v_i^\varepsilon - v_i^0\|_{L^2(\Gamma^\varepsilon)}^2 \right) \\ &\quad + \varepsilon \int_{\Gamma^\varepsilon} |a_i^\varepsilon - A_i| |u_i^0| |v_i^\varepsilon - v_i^0| dS_\varepsilon + \varepsilon \int_{\Gamma^\varepsilon} |b_i^\varepsilon - B_i| |v_i^0| |v_i^\varepsilon - v_i^0| dS_\varepsilon. \end{aligned} \quad (4.21)$$

Since Ω^ε is a Lipschitz domain, we recall the trace embedding $H^1(\Omega^\varepsilon) \subset L^q(\partial\Omega^\varepsilon)$ which holds for $1 \leq q \leq 2_{\partial\Omega^\varepsilon}^*$ where $2_{\partial\Omega^\varepsilon}^* = 2(d-1)/(d-2)$ if $d \geq 3$, and $2_{\partial\Omega^\varepsilon}^* = \infty$ if $d = 2$ (cf. [13]). Therefore, when the two-dimensional case is concentrated, we continue to estimate (4.21), as follows:

$$\begin{aligned} \frac{\varepsilon}{2} \frac{d}{dt} \sum_{i=1}^N \|v_i^\varepsilon - v_i^0\|_{L^2(\Gamma^\varepsilon)}^2 &\leq C\varepsilon \left(\sum_{i=1}^N \|u_i^\varepsilon - u_i^0\|_{L^2(\Gamma^\varepsilon)}^2 + \sum_{i=1}^N \|v_i^\varepsilon - v_i^0\|_{L^2(\Gamma^\varepsilon)}^2 \right) \\ &\quad + C\varepsilon \left(\sum_{i=1}^N \|a_i^\varepsilon - A_i\|_{L^2(\Gamma^\varepsilon)}^2 + \sum_{i=1}^N \|b_i^\varepsilon - B_i\|_{L^2(\Gamma^\varepsilon)}^2 \right). \end{aligned}$$

Observe that using the trace inequality (2.34) for the difference norms $\|a_i^\varepsilon - A_i\|_{L^2(\Gamma^\varepsilon)}$, $\|b_i^\varepsilon - B_i\|_{L^2(\Gamma^\varepsilon)}$ and $\|u_i^\varepsilon - u_i^0\|_{L^2(\Gamma^\varepsilon)}$ together with Lemma 14 and (4.19) gives

$$\frac{\varepsilon}{2} \frac{d}{dt} \sum_{i=1}^N \|v_i^\varepsilon - v_i^0\|_{L^2(\Gamma^\varepsilon)}^2 \leq C \max\{\varepsilon, \varepsilon^\gamma\} + C\varepsilon \sum_{i=1}^N \|v_i^\varepsilon - v_i^0\|_{L^2(\Gamma^\varepsilon)}^2. \quad (4.22)$$

Note herein that the gradient norms are ignored when applying the trace inequality to the differences. It is simply because that they are of the order $\mathcal{O}(\varepsilon^2)$ by their own regularity.

Henceforward, we apply the Gronwall inequality to (4.22) and obtain

$$\varepsilon \sum_{i=1}^N \|v_i^\varepsilon - v_i^0\|_{L^2(\Gamma^\varepsilon)}^2 \leq C \max\{\varepsilon, \varepsilon^\gamma\} e^{C\varepsilon t}.$$

In the same manner, if $d \geq 3$ is applied, we can bound the absolute differences $|a_i^\varepsilon - A_i|$ and $|b_i^\varepsilon - B_i|$ in (4.21) from above by a constant C independent of ε (by (A_1)) and then get back the estimate (4.22).

This completes the proof of Theorem 12.

5 Conclusions

In this work, we have presented corrector estimates for the homogenization limit for a thermo-diffusion system with Smoluchowski interactions coupled with a system of differential equations, posed in a perforated domain. This type of error-control justifies the formal homogenization asymptotics obtained in [25] and completes the convergence result in [24] by giving convergence rates. This is done using the concept of macroscopic reconstruction together with fine integral estimates on the solution and oscillating coefficients. Our working technique can be applied to a larger class of coupled nonlinear systems of partial differential equations posed in perforated media.

Acknowledgment

This work was initiated when V.A.K. visited the Department of Mathematics and Computer Science of Karlstad University, Sweden. This work is dedicated to the memory of his beloved father. A.M. thanks NWO MPE ‘Theoretical estimates of heat losses in geothermal wells’ (grant nr. 657.014.004) for funding.

References

- [1] R.A. Adams. *Sobolev Spaces*. Academic Press, New York, 1975.
- [2] G. Allaire. Homogenization and two-scale convergence. *SIAM Journal on Mathematical Analysis*, 23:1482–1518, 1992.
- [3] G. Allaire, A. Damlamian, and U. Hornung. Two-scale convergence on periodic surfaces and applications. In *Proc. International Conference on Mathematical Modelling of Flow through Porous Media*. World Scientific, 1995.
- [4] B. Amaziane and L. Pankratov. Homogenization of a reaction-diffusion equation with Robin interface conditions. *Applied Mathematics Letters*, 19:1175–1179, 2006.
- [5] D. Ciorănescu and P. Donato. *An Introduction to Homogenization*. Oxford Lecture Series in Mathematics and Its Applications, 1999.
- [6] D. Ciorănescu and J. Saint Jean Paulin. *Homogenization of Reticulated Structures*. Springer, 1999.
- [7] C. Le Bris, F. Legoll, and A. Lozinski. An MsFEM type approach for perforated domains. *SIAM Multiscale Modeling and Simulation*, 12(3):1046–1077, 2014.
- [8] C. Eck. Analysis of a two-scale phase field model for liquid-solid phase transitions with equiaxed dendritic microstructure. *Multiscale Modeling and Simulation*, 3(1):28–49, 2004.

- [9] C. Eck. Homogenization of a phase field model for binary mixtures. *Multiscale Modeling and Simulation*, 3:1–27, 2004.
- [10] M. Eden and A. Muntean. Corrector estimates for the homogenization of a two-scale thermoelasticity problem with a priori known phase transformations. *Electronic Journal of Differential Equations*, 2017(57):1–21, 2017.
- [11] M. Elimelech, J. Gregory, X. Jia, and R. Williams. *Particle Deposition & Aggregation: Measurement, Modelling and Simulation*. Elsevier, 1998.
- [12] L. Evans. *Partial Differential Equations*, volume 19. American Mathematical Society, 1998.
- [13] X. Fan. Boundary trace embedding theorems for variable exponent Sobolev spaces. *Journal of Mathematical Analysis and Applications*, 339:1395–1412, 2008.
- [14] T. Fatima. and A. Muntean. Sulfate attack in sewer pipes: Derivation of a concrete corrosion model via two-scale convergence. *Nonlinear Analysis: Real World Applications*, 15:326–344, 2014.
- [15] T. Fatima, A. Muntean, and M. Ptashnyk. Unfolding-based corrector estimates for a reaction-diffusion system predicting concrete corrosion. *Applicable Analysis*, 91(6):1129–1154, 2012.
- [16] A.K. Giri, J. Kumar, and G. Warnecke. The continuous coagulation equation with multiple fragmentation. *Journal of Mathematical Analysis and Applications*, 374(1):71–87, 2011.
- [17] A.K. Giri and G. Warnecke. Uniqueness for the coagulation-fragmentation equation with strong fragmentation. *Zeitschrift für Angewandte Mathematik und Physik*, 62(6):1047–1063, 2011.
- [18] G. Griso. Error estimate and unfolding for periodic homogenization. *Asymptotic Analysis*, 40:269–286, 2004.
- [19] P. Grisvard. *Elliptic Problems in Nonsmooth Domains*. Pitman, London, 1985.
- [20] U. Hornung and W. Jäger. Diffusion, convection, adsorption, and reaction of chemicals in porous media. *Journal of Differential Equations*, 92:199–225, 1991.
- [21] V.A. Khoa. A high-order corrector estimate for a semi-linear elliptic system in perforated domains. *Comptes Rendus Mécanique*, 2017. to appear, DOI: 10.1016/j.crme.2017.03.003.
- [22] V.A. Khoa and A. Muntean. Asymptotic analysis of a semi-linear elliptic system in perforated domains: Well-posedness and corrector for the homogenization limit. *Journal of Mathematical Analysis and Applications*, 439:271–295, 2016.
- [23] S.M. Kozlov. Averaging differential operators with almost periodic, rapidly oscillating coefficients. *Mathematics of the USSR-Sbornik*, 35:481–498, 1979.
- [24] O. Krehel, T. Aiki, and A. Muntean. Homogenization of a thermo-diffusion system with Smoluchowski interactions. *Networks and Heterogeneous Media*, 9(4):739–762, 2014.
- [25] O. Krehel, A. Muntean, and P. Knabner. Multiscale modeling of colloidal dynamics in porous media including aggregation and deposition. *Advances in Water Resources*, 86:209–216, 2015.
- [26] O. Lakkis and C. Makridakis. Elliptic reconstruction and a posteriori error estimate for fully discrete linear parabolic problems. *Mathematics of Computation*, 75(256):1627–1658, 2006.
- [27] A. Muntean and S. Reichelt. Corrector estimates for a thermo-diffusion model with weak thermal coupling. WIAS, Preprint No. 2310, 2016.

- [28] A. Muntean and T.L. van Noorden. Corrector estimates for the homogenization of a locally-periodic medium with areas of low and high diffusivity. *European Journal of Applied Mathematics*, 24(5):657–677, 2012.
- [29] M. Neuss-Radu. Some extensions of two-scale convergence. *Comptes Rendus de l Académie des Sciences-Series I-Mathematics*, 352:899–904, 1996.
- [30] G. Nguetseng. A general convergence result for a functional related to the theory of homogenization. *SIAM Journal on Mathematical Analysis*, 20:608–623, 1989.
- [31] D. Onofrei and B. Vernescu. Error estimate and unfolding for periodic homogenization with non-smooth coefficients. *Asymptotic Analysis*, 54:103–123, 2007.
- [32] L.E. Persson, L. Persson, N. Svanstedt, and J. Wyller. *The Homogenization Method: An Introduction*. Chartwell Bratt, Sweden, 1993.
- [33] A.V. Pozhidaev and V.V. Yurinskii. On the error of averaging symmetric elliptic systems. *Mathematics of the USSR-Izvestiya*, 35:183–201, 1990.
- [34] N. Ray, A. Muntean, and P. Knabner. Rigorous homogenization of a Stokes-Nernst-Planck-Poisson system. *Journal of Mathematical Analysis and Applications*, 390(1):374–393, 2012.
- [35] G. Savaré. Regularity results for elliptic equations in Lipschitz domains. *Journal of Functional Analysis*, 152:176–201, 1998.
- [36] M. Schmuck, G.A. Pavliotis, and S. Kalliadasis. Effective macroscopic interfacial transport equations in strongly heterogeneous environments for general homogeneous free energies. *Applied Mathematics Letters*, 35:12–17, 2014.
- [37] M. Smoluchowski. Versuch einer mathematischen Theorie der Koagulationskinetik kolloider Lösungen. *Zeitschrift für Physikalische Chemie*, 92:129–168, 1917.